Laminar boundary layer

Why B.L. theory?

- With the potential theory only pressure distributions can be determined
  \[ \Rightarrow \text{lift force} \]

- The drag force cannot be described with the potential theory. Viscous forces have to be considered.

The pressure distribution at slender bodies fits well with the theoretical distribution from the potential theory, if \( Re \gg 1 \). The influence of the viscous forces is limited to a thin layer near the wall. \( \Rightarrow \) boundary layer

Example:

![Diagram of laminar boundary layer]
• The segmentation of the flow into two parts (the frictionless outer part and the viscous boundary layer) allows a complete description of the flow field.

• The boundary layer theory is not valid in nose regions.

• Due to the deceleration in the boundary layer, the streamlines are pushed away from the wall. The streamlines are no longer parallel to the wall.

• The line $s(x)$, describing the edge of the boundary layer (i.e., thickness) is not a streamline. It denotes the line where the velocity reaches the value of the outer flow up to a certain amount. Usually 91%:

$$\frac{U(x)}{U_a} = 0.99$$ arbitrary
Approximation of the boundary layer thickness

In the boundary layer:

\[ o(\text{Inertia}) \propto o(\text{Viscous forces}) \]
\[ \frac{\rho u}{\rho} \frac{\partial u}{\partial x} \approx \frac{\partial^2 u}{\partial y^2} \]

Dimensionless values \( \rightarrow O(1) \)

\[ \bar{u} = \frac{u}{u_\infty}, \quad \bar{x} = \frac{x}{L}, \quad \bar{y} = \frac{y}{S} \]

\[ \bar{u} \approx \frac{u_\infty}{L} \approx \frac{U_\infty}{S} \]

\[ \bar{S} \approx \frac{1}{\sqrt{\frac{u_\infty S}{L}}} = \frac{1}{\sqrt{\sqrt{\frac{L}{u_\infty}}} \sqrt{\frac{L}{S}}} \]

Introduction of dimensionless variables in the Navier-Stokes equations for 2-d, steady, incompressible flows. Dimensionless variables are \( O(1) \).

Neglect all terms with the factor \( \frac{1}{Re} \) or smaller

\[ \Rightarrow \text{Boundary layer equations are valid for } Re \gg 1 \]
Continuity: \( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \)

x-mom: \( su \frac{\partial u}{\partial x} + sv \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \xi \frac{\partial^2 u}{\partial y^2} \)

y-mom: \( \frac{\partial p}{\partial y} = 0 \)

y-mom: The pressure is constant normal to the main stream direction. It is imposed from the frictionless outer flow.

Boundary layer edge for a flat plate (\( y = \delta \))

\[ u = U \Rightarrow \frac{\partial u}{\partial y} = 0 \Rightarrow \frac{\partial^2 u}{\partial y^2} = 0 \]

x-mom: \( s u \frac{\partial u}{\partial x} = -\frac{\partial p}{\partial x} \]

Euler-equation for \( y = \delta \)

3 cases depending on the pressure gradient

1. \( \frac{\partial p}{\partial x} = 0 = \frac{\partial u}{\partial x} = 0 \Rightarrow U = \text{const} \)

Flat plate

boundary layer

(Blasius)
2. \( \frac{\partial P}{\partial x} < 0 \Rightarrow \frac{\partial U}{\partial x} > 0 \Rightarrow \text{accelerated flow} \)

Convergent channel (nozzle)

\[ U(x) \rightarrow \quad \text{at} \quad x = 2 \quad U_x < U_2 \]

3. \( \frac{\partial P}{\partial x} > 0 \Rightarrow \frac{\partial U}{\partial x} < 0 \Rightarrow \text{decelerated flow} \)

Divergent channel (diffuser)

\[ U(x) \rightarrow \quad \text{at} \quad x = 2 \quad U_x > U_2 \]

Separation

At the wall (\( y = 0 \))

No-slip condition: \( U = v = 0 \)

\( \Rightarrow \) x-momentum: \( \frac{\partial P}{\partial x} = \gamma \frac{\partial^2 U}{\partial y^2} \Big|_{y=0} \)

\[ \Rightarrow \frac{\partial P}{\partial x} = \frac{\partial U}{\partial y} \Big|_{y=0} \]

Flat plate: \( \frac{\partial P}{\partial x} = 0 \) (no pressure gradient, \( U = \text{constant} \))

\( \Rightarrow \frac{\partial^2 U}{\partial y^2} \Big|_{y=0} = 0 \) (\( \frac{\partial U}{\partial y} \Big|_{y=0} = \text{constant} \))

\( \Rightarrow \) no curvature of the velocity profile at the wall
$s_n$ (displacement-thickness): characteristic measure for the displacement of an undisturbed streamline.

Areas are the same

\[
\text{mass flux width} = \text{constant}
\]

\[
S_n \quad \text{from } u = \text{constant}.
\]

\[
U (s - s_n) = \int S \, dy \quad \Rightarrow \quad \int S \, u - u \, dy = U \, s_n
\]

\[
S_n = \int_0^S \left( 1 - \frac{U(y)}{U} \right) \, dy
\]

in dimensional form

\[
\frac{s_n}{S} = \int_0^S \left( 1 - \frac{U(y)}{U} \right) \, d \frac{Y}{S} \quad (S = f(Y))
\]
Due to friction some momentum losses occur against the undisturbed flow.

From a momentum balance:

\[
S_2 = \int_0^\infty \frac{U}{U} \left(1 - \frac{U}{U}\right) d\gamma
\]

\[
\frac{S_2}{8} = \int_0^1 \frac{U}{U} \left(1 - \frac{U}{U}\right) d\left(\gamma^2\right)
\]

To compute the drag, these two measures + the von Kármán - Integral - equation is used.

Integration of the x - momentum equation

\[
\frac{d}{dx} \left(U^2 S_2 \right) + \xi_n U \frac{dU}{dx} = \frac{\tau_w}{8}
\]

or

\[
\frac{dS_2}{dx} + \frac{1}{U} \frac{dU}{dx} \left(2S_2 + \xi_n\right) = \frac{\tau_w}{8U^2}
\]

- Assume a polynomial for the velocity.
- Use the boundary conditions to compute the coefficients.
- Compute \(\xi_n\) and \(S_2\).
- Use the von Kármán - equation to compute \(T_w(x)\) or \(S(x)\).
Assumption: polynomial profile
\[ \frac{u(x,y)}{U(x)} = \sum_{i=0}^{n} a_i \left( \frac{y}{\delta} \right)^i = f(x, \frac{y}{\delta}) \]

self-similar profile: \( a_i(x), f(x) \)

boundary conditions
1) no-slip condition (stokes) for \( \frac{y}{\delta} = 0 \Rightarrow u = v = 0 \) \((u_\delta = U_\infty)\)

2) B.L. edge \( \frac{y}{\delta} = 1 = u = U \)

3) from x-momentum
\[ \frac{\partial u}{\partial y} \bigg|_{y=0} = \frac{\partial p}{\partial x} \left( = 0 \text{ for a flat plate} \right) \]

\( \frac{\partial p}{\partial x} \) from Euler-equation Bernoulli:

Only, if the degree of the polynomial is \( \geq 2 \), other boundary conditions are necessary.

from the continuity at the B.L. edge

4) \( \frac{y}{\delta} > 1 \Rightarrow \frac{\partial u}{\partial y} = 0 \)

continuous transition from B.L. to outer flow

5) \( \frac{y}{\delta} = 1 \Rightarrow \frac{\partial^2 u}{\partial y^2} = 0 \)

frictionless flow
Remark: \( u_a = U_a = U = U_e = \ldots \) different notations

a) \( a_0, a_n = ? \)

- no-slip condition: \( u(y=0) = 0 \rightarrow a_0 = 0 \)
- at the b.l. edge: \( \frac{u}{U} \bigg|_{y/b_o = 1} = 1 \rightarrow a_n = 1 \)

\[
\Rightarrow \frac{u(x,y)}{U(x)} = \frac{y}{b_o}
\]

b) displacement thickness: \( \frac{\delta_n}{b_o} = \int_0^1 \left( 1 - \frac{y}{U} \right) d\frac{y}{b_o} \)

\[
= \left[ \frac{y}{b_o} - \frac{1}{2} \left( \frac{y}{b_o} \right)^2 \right]_0^1 = \frac{1}{2} \rightarrow \delta_n = \frac{1}{2} b_o
\]

- momentum thickness: \( \frac{\delta_2}{b_o} = \int_0^1 \frac{u}{U} \left( 1 - \frac{y}{U} \right) d\frac{y}{b_o} \)

\[
= \frac{1}{2} \left( \frac{y}{b_o} \right)^2 - \frac{1}{3} \left( \frac{y}{b_o} \right)^3 \bigg|_0^1 = \frac{1}{6} \rightarrow \delta_2 = \frac{1}{6} b_o
\]

\[
\frac{\partial \delta_2}{\partial \delta_0} = \frac{1}{6} = \text{const} \rightarrow \frac{\partial \delta_2}{\partial x} = 0
\]

wall shear stress: \( \tau_w = \frac{1}{2} \frac{\partial u}{\partial y} \bigg|_{y=a} = 0 \)

\[
\tau_w = \frac{1}{2} \frac{U}{b_o} \frac{\partial u(y/b_o)}{\partial x} = \frac{1}{2} \frac{U}{b_o}
\]
von Kármán equation

\[
\frac{dU}{dx} = U \cdot \frac{1}{8U^2} \cdot \frac{U}{S_0^2} \left( \frac{1}{2 \cdot \frac{1}{6} + \frac{1}{2}} \right) = \frac{2}{8S_0^2} \cdot \frac{6}{5}
\]

\[U(x) = \frac{7}{8S_0^2} \cdot \frac{6}{5} \times (\text{Usually} \; S = S(x))\]

3) \[F = \int_0^L C(x) \cdot B \cdot dx\]

\[= \frac{6}{5} \cdot \frac{7}{8S_0^2} \cdot \frac{2}{S_0} \cdot B \int_0^L dx\]

\[= \frac{12}{5} \cdot \frac{7}{8S_0^2} \cdot \frac{B L^2}{S_0^3}\]
15.4 The velocity profile of a laminar incompressible boundary layer with constant viscosity $\eta$ can be described with a polynomial:

$$\frac{u(x, y)}{u_a(x)} = a_0 + a_1(x) \left( \frac{y}{\delta} \right) + a_2(x) \left( \frac{y}{\delta} \right)^2 + a_3(x) \left( \frac{y}{\delta} \right)^3$$

The outer velocity $u_a(x)$ is given with the following approach:

$$u_a(x) = u_{a1} - C \cdot (x - x_1)^2$$

$u_{a1}$ is the outer velocity at $x_1$ and $C$ is a positive constant. The boundary layer thickness at $x_2$ is $\delta(x_2)$.

Given: $\rho, \eta, x_1, u_{a1}, \delta(x_2), C$, with $C > 0$

Determine:

a) the pressure gradient $\partial p/\partial x$ in the flow as a function of $x$.

b) the coefficient $a_0$ and the coefficients $a_1(x), a_2(x), a_3(x)$.
\[\frac{\partial p}{\partial x} = ?\]

Frictionless outflow gives:

x-mom 
\[g \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = - \frac{\partial p}{\partial x} + \rho \frac{\partial^2 u}{\partial y^2}\]

at the b.l. edge: \(\frac{\partial u}{\partial y} = 0\); \(\frac{\partial^2 u}{\partial y^2} = 0\)

\[\Rightarrow \quad g u \frac{\partial u}{\partial x} = - \frac{\partial p}{\partial x}\]

\[u = u_0 - c (x-x_n)^2\]

\[\Rightarrow \quad \frac{\partial u}{\partial x} = -2c (x-x_n)\]

\[\Rightarrow \quad \frac{\partial p}{\partial x} = 2cg \left[ u_0 (x-x_n) - c (x-x_n)^3 \right]\]

\[\frac{\partial p}{\partial x} \bigg|_{x=x_n} = 0\]

\[4 \text{ coefficients} \rightarrow 4 \text{ boundary conditions}\]

1.) No-slip condition: \(\frac{Y}{S} = 0 \Rightarrow u = 0 \Rightarrow a_0 = 0\)

2.) b.l. edge: \(\frac{Y}{S} = 1\): \(u = U\)

\[\Rightarrow \quad a_0 + a_1 + a_2 + a_3 = 1\]

3.) at the wall: \(\frac{Y}{S} = 0 = g \frac{\partial^2 u}{\partial y^2} \bigg|_{y=0} = \frac{\partial p}{\partial x}\)
\[ x = \left[ a_1 \left( \frac{y}{\delta} \right) + a_2 \left( \frac{y}{\delta} \right)^2 + a_3 \left( \frac{y}{\delta} \right)^3 \right] \]

\[ \frac{\partial a}{\partial y} = \left( \frac{a_1}{\delta} + 2 \frac{a_2}{\delta^2} y + 3 a_3 / \delta^3 y^3 \right) u \]

\[ \frac{\partial^2 a}{\partial y^2} = \left( 0 + 2 \frac{a_2}{\delta^2} + 6 a_3 \frac{y}{\delta^3} \right) u \]

\[ Y_{x=0} \Rightarrow \frac{\partial p}{\partial x} = \eta \left( \frac{a_2}{\delta^2} u \right) \Rightarrow a_2(x) = \frac{1}{2} \frac{\delta^2}{\eta} \frac{\partial p}{\partial x} \]

\[ a_2(x) = \frac{\delta^2}{\eta} \frac{2 \eta C (x-x_n) (U - C (x-x_n)^2)}{U - C (x-x_n)^2} \]

\[ \Rightarrow a_2(x) = \frac{\delta^2 C}{\eta} (x-x_n) \]

4.) Continuous distribution at b.c. edge

\[ \frac{y}{\delta} = \lambda \Rightarrow \frac{\partial a}{\partial y} = 0 \Rightarrow \frac{a_1}{\delta} + 2 \frac{a_2}{\delta^2} + 3 \frac{a_3}{\delta^3} = 0 \]

\[ \Rightarrow a_1 + a_2 + a_3 = 1 \]

\[ a_3 = -\frac{1}{2} (a_1 + a_2) \]

\[ a_1 + 2a_2 + 3a_2 = 0 \]

\[ a_1 = \frac{1}{2} (3-a_1) \]

\[ a_1(x) = \frac{3}{2} - \frac{1}{2} \frac{\delta^2 C}{\eta} (x-x_n) \]

\[ a_2(x) = \frac{1}{2} - \frac{1}{2} \frac{\delta^2 C}{\eta} (x-x_n) \]

\[ a_3(x) = -\frac{1}{2} - \frac{1}{2} \frac{\delta^2 C}{\eta} (x-x_n) \]