Hydrodynamics

Continuity equation

Conservation of mass: \[ \dot{m}_1 = \dot{m}_2 \]

\[ \frac{\rho_1 v_1 A_1}{\dot{m}_1} = \frac{\rho_2 v_2 A_2}{\dot{m}_2} \]

inkompressible fluid: \( \rho_1 = \rho_2 = \text{const} \)

Conservation of volume flux: \[ \dot{Q}_1 = \dot{Q}_2 \]

\[ \frac{v_1 A_1}{\dot{Q}_1} = \frac{v_2 A_2}{\dot{Q}_2} \]
Hydrodynamics

Example pipe flow: $A = const$

![Closed stream tube diagram]

Water jet

![Flow diagram with mass flow rates]

Closed control volume

$m_1 = m_2 + m_3$
Important: In the 1-dimensional continuity equation $\vec{v}$ is an average value of the velocity. In reality $\vec{v}$ is not constant due to friction, vortices, ...!

In reality $\vec{v} = \vec{v}(y)$

$v$ ist constant in the continuity equation

Mass flux must be constant

$$\int \rho v(y) \, dy = \rho \bar{v} h$$
2. Newtonian law:
mass $\times$ acceleration = sum of outer forces

$$m \cdot \frac{d\vec{v}}{dt} = \sum F_a$$

Equation of motion for an infinitesimal element along one streamline

$$\rho \frac{d\vec{v}}{dt} = -\frac{\partial p}{\partial s} - \rho g \frac{dz}{ds} - R'$$

inertia
pressure
friction
gravitation
Bernoulli

along a streamline: \( v = v(s, t) \)

\[ d\vec{v} = \frac{\partial \vec{v}}{\partial t} dt + \frac{\partial \vec{v}}{\partial s} ds \]

\[ \frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \frac{ds}{dt} \frac{\partial \vec{v}}{\partial s} = \frac{\partial \vec{v}}{\partial t} + v \frac{\partial \vec{v}}{\partial s} \]

total (substantial) acceleration

lokale acceleration | convective acceleration

of a particle
$A, \rho = \text{konst}$

$\rightarrow v_1(t) = v_2(t)$

only local acceleration  
only convective acceleration
example

assumptions:

- incompressible \((\rho = \text{const})\)
- frictionless \((R' = 0)\)
- steady \(\frac{\partial}{\partial t} = 0\)
- constant gravitation \((\vec{g} = \text{const})\)

\[
\rho \left[ \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial s} \right] = -\frac{\partial p}{\partial s} - \rho g \frac{dz}{ds} - R'
\]

\[
= 0 = 0
\]

\[
f(s) \rightarrow \frac{\partial}{\partial s} = \frac{d}{ds}
\]

\[
\frac{1}{2} \rho \frac{dv^2}{ds} = -\frac{dp}{ds} - \rho g \frac{dz}{ds} \rightarrow \frac{\rho}{2} v^2 + p + \rho gz = \text{const}
\]
**pressure measurement**

**static pressure:** \( p \) (Index: 1, 2, \( a \), \( \infty \))

\[
p_1 \quad \rightarrow \quad p
\]

**Total pressure (pitot tube):** \( p_0, p_{01}, p_{02}, p_t \)

\[
p_0 = p + \frac{1}{2}\rho v^2 + \rho gh
\]

at constant height \( \Delta h = 0 \)

\[
\rightarrow p_0 = p + \frac{1}{2}\rho v^2
\]
potential pressure: \( p_{pot} = \rho gh \)

kinetic energy is converted, when the flow is decelerated to \( \vec{v} = 0 \)
Water flows from a large pressurized tank into the open air. The pressure difference $\Delta p$ is measured between $A_1$ and $A_2$.

\[ A_1 = 0.3 \, m^2, \quad A_2 = 0.1 \, m^2, \]
\[ A_3 = 0.2 \, m^2, \quad h = 1 \, m, \]
\[ \rho = 10^3 \, kg/m^3, \quad p_a = 10^5 \, N/m^2, \]
\[ \Delta p = 0.64 \cdot 10^5 \, N/m^2, \quad g = 10 \, m/s^2. \]

Compute the
a) velocities $v_1, v_2, v_3$,
b) pressures $p_1, p_2, p_3$ and the pressure $p$ above the water surface!
pressure tank with nozzle

$P_B$

$h = \text{konst.}$

gut gerundeter Einlass

Venturidüse
conservation of total energy along a streamline $\rightarrow$ qualitatively

Bernoulli: $p_0 = p_b + \rho gh = p_i + \frac{1}{2} \rho v_i^2$
continuity (mass balance): \[ \Rightarrow m = \rho \dot{Q} = \text{const.} \]

\[ \rho = \text{const} \Rightarrow v_1 A_1 = v_2 A_2 = v_3 A_3 \Rightarrow A \downarrow \Rightarrow v \uparrow \Rightarrow p \downarrow \]

a) measured \[ \Delta p = p_1 - p_2 \] 

Bernoulli: \[ p_1 + \frac{\rho}{2} v_1^2 = p_2 + \frac{\rho}{2} v_2^2 \]

\[ \Rightarrow \Delta p = p_1 - p_2 = \frac{\rho}{2}(v_2^2 - v_1^2) > 0 \]

\[ v_1 = v_2 \frac{A_2}{A_1} \rightarrow \Delta p = \frac{\rho}{2} \left[ 1 - \frac{A_2^2}{A_1^2} \right] v_2^2 \rightarrow v_2 = \sqrt{\frac{2}{\rho \left( 1 - \left( \frac{A_2}{A_1} \right)^2 \right)}} \Delta p = 12 \frac{m}{s} \]

\[ v_1 = v_2 \frac{A_2}{A_1} = 4 \frac{m}{s} \]

\[ v_3 = v_2 \frac{A_2}{A_3} = 6 \frac{m}{s} \]
The Venturi-nozzle is used to measure mass- and volume fluxes!

\[ \dot{Q} = vA = v_2A_2 \]

prinziple:

- measurement of \( \Delta p \)
- computation of \( v_2 \)
- computation of volume- and massflux
b) determination of pressures $p_B, p_1, \ldots, p_3$

$p_0$ represents the energy that can be converted into kinetic energy

$$p_0 = p_B + \rho gh = p_1 + \frac{\rho}{2} v_1^2 = p_2 + \frac{\rho}{2} v_2^2 = p_3 + \frac{\rho}{2} v_3^2$$

If we know one pressure, we can compute the other values by using Bernoulli’s equation

$p_3$ in the exit cross section

**Assumption:** parallel streamlines at the sharp edged exit
equation of motion for an element

equation of motion in $x$-direction for a moving control volume $dA\,dx$ (includes always the same particles!)
equation of motion for an element

\[ m \frac{du}{dt} = \ddot{x} \rho dA \, dx = p(x) \, dA - p(x + dx) \, dA \]

\[ \Rightarrow \quad \ddot{x} \rho dA \, dx = p(x) \, dA - \left( p + \frac{\partial p}{\partial x} \, dx \right) \, dA \]

\[ \Rightarrow \quad \rho \ddot{x} = -\frac{\partial p}{\partial x} \]

Assumption: parallel stream lines

\[ \Rightarrow \quad \dot{x} = 0 \quad \text{velocity} \quad u = \frac{dx}{dt} = \dot{x} \]

\[ \Rightarrow \quad \text{necessary condition:} \quad \ddot{x} = 0 \quad \Rightarrow \quad \frac{\partial p}{\partial x} = 0 \]

\[ \Rightarrow \quad \text{the pressure in the exit cross-section is function of } y \]

flow into air: \[ \frac{dp}{dy} = -\rho g \]

Neglect the potential energy \[ p_{\text{exit}} = p_{\text{ambience}} = \text{const.} \]
\[ p_3 = p_a \]

Remark:

Bernoulli: \( 0 \rightarrow 3 \)

\[ p_B + \rho gh = p_a + \frac{1}{2} \rho v_3^2 \]

\[ \rightarrow v_3 = \sqrt{\frac{2}{\rho} (p_B - p_a + \rho gh)} \]

open tank \( p_B = p_a \)

\[ \rightarrow v_3 = \sqrt{2gh} \neq f(A_3) \]

theorem of Torricelli (15.Okt. 1608 - 25.Okt. 1647)
Two large basins located one upon the other are connected with a duct.

\[ A = 1 \, m^2, \quad A_d = 0,1 \, m^2, \quad h = 5 \, m, \quad H = 80 \, m, \]
\[ p_a = 10^5 \, N/m^2, \quad \rho = 10^3 \, kg/m^3, \quad g = 10 \, m/s^2 \]

a) Determine the volume rate!
b) Outline the distribution of static pressure in the duct!
c) At what exit cross section bubbles are produced, when the vapour pressure is
\[ p_D = 0,025 \cdot 10^5 \, N/m^2? \]
incompressible, frictionless, steady
\[ \rightarrow \text{Bernoulli} \]
a) volume flux: \[ \dot{Q} = vA = v_5A_5 \]
Bernoulli: \[ 0 \rightarrow 5 \]

\[ p_a + \rho g H = p_5 + \rho g(-s) + \frac{1}{2}v_5^2 \]

\[ \rightarrow p_5 = p_a + \rho gs \]
\[ p_a + \rho g H = p_a + \rho g s - \rho g s + \frac{1}{2} \rho v_5^2 \quad \rightarrow \quad v_5 = \sqrt{2gH} \neq f(A_d, s) \]

\[ \rightarrow \dot{Q} = A_d v_5 = 4 \frac{m^3}{s} \]
c) minimum pressure between 2 and 3: \( p_2 = p_3 = p_D \)

continuity: \( v_5^* A_D^* = v^* A \)

Bernoulli: \( p_a = p_D + \rho g h + \frac{1}{2} \rho v^*^2 \)

\[
\rightarrow A_d = A \sqrt{\frac{p_a - p_D}{\rho g H}} - \frac{h}{H} = 0.244 \text{ m}^2
\]
extended Bernoulli

\[ \rho = \text{const.} \]

\[ \Delta h \sim \Delta p \]

Theoretical volume flux: \( \dot{Q}_{th} \) for frictionless flow

1. Bernoulli: \( p_1 + \rho \frac{v_1^2}{2} = p_2 + \rho \frac{v_2^2}{2} \)
2. continuity: \( v_1 A_1 = v_2 A_2 \)
extended Bernoulli

ratio of areas: \( m = \frac{A_2}{A_1} \): \( \longrightarrow \) conti \( v_1 = v_2 m \)

\[ \longrightarrow \text{ Bernoulli: } \frac{p_1}{\rho} + \frac{1}{2}v_2^2 m^2 = \frac{p_2}{\rho} + \frac{1}{2}v_2^2 \]

\[ \longrightarrow v_2^2 \left(1 - m^2\right) = 2\frac{p_1 - p_2}{\rho} = 2\frac{\Delta p}{\rho} \]

\[ \longrightarrow v_2 = \left[\frac{2\Delta p}{\rho(1 - m^2)}\right]^{\frac{1}{2}} \]

\[ \longrightarrow \dot{Q}_{th} = A_2 \left[\frac{2\Delta p}{\rho(1 - m^2)}\right]^{\frac{1}{2}} \]
extended Bernoulli (Cont’d)

In reality losses from friction, vortices, bottle necks . . . occur. → the flow is no longer frictionless

The losses and the ration of area are put together in the discharge coefficient $\alpha$

\[
\dot{Q}_{\text{real}} = \alpha A_2 \sqrt{\frac{2\Delta p}{\rho(1 - m^2)}}
\]

$\alpha^* = \alpha \sqrt{\frac{1}{1 - m^2}}$

$\alpha^*$ from experiments

vortex, dissipation

losses in pipe flows can be predicted similar.
extended Bernoulli (Cont’d)

pressure loss across a constructive element (ellbow, valve, …)

\[ \Delta p_v = \zeta \cdot \frac{1}{2} \rho v^2 \]

coeffizient: \( \zeta = \frac{\Delta p_v}{\frac{1}{2} \rho v^2} = \frac{\text{pressure loss}}{\text{dynamic pressure}} \)

\[ \rightarrow v = \frac{1}{\sqrt{\zeta}} \sqrt{\frac{2\Delta p}{\rho(1 - m^2)}} \implies \dot{Q} = v \cdot A = \frac{1}{\sqrt{\zeta}} A \sqrt{\frac{2\Delta p}{\rho(1 - m^2)}} \]

(Experiments, standards \(\rightarrow\) catalogue)
Hydrodynamics: unsteady Bernoulli

assumption
\[ \frac{A_1}{A_0} \ll 1 \quad \text{→} \quad v_0 \ll v_1 \]

\( v_0 \) is neglectable
but \( h = h(t) \) and \( v_1 = v_1(t) \)

unsteady Bernoulli from "0" to "1"

\[
p_a + \frac{\rho}{2}v_0^2(t) + \rho gh(t) = p_a + \frac{\rho}{2}v_1^2(t) + \int_0^1 \rho \frac{\partial v}{\partial t} ds
\]

Assumption \( v_1(t) = \sqrt{2gh(t)} \)

continuity equation: \( v_1(t)A_1 = -\frac{dh}{dt}A_0 \)

→ differential equation for \( h(t) \)
Hydrodynamics: unsteady Bernoulli

“well rounded” inlet

assumption:

\[ s < -\frac{D}{\sqrt{8}} : \text{radial flow with } \dot{Q} = v \cdot \frac{\pi s^2}{2} \]

\[ s \geq -\frac{D}{\sqrt{8}} : v = v_1 \]
Hydrodynamics: unsteady Bernoulli

Potential flow without any losses $\rightarrow$ Bernoulligleichung

$$\int_{-\infty}^{L} \frac{\partial v}{\partial t} ds = \int_{-\infty}^{-D/\sqrt{8}} \frac{\partial v(s)}{\partial t} ds + \int_{-D/\sqrt{8}}^{L} \frac{\partial v_1}{\partial t} ds = \int_{-\infty}^{-D/\sqrt{8}} \frac{\partial}{\partial t} \left( \frac{v_1 \pi D^2}{4} \right) ds + \int_{-D/\sqrt{8}}^{L} \frac{\partial v_1}{\partial t} ds$$

$$= \frac{dv_1(t)}{dt} \int_{-\infty}^{-D/\sqrt{8}} \frac{D^2}{8s^2} ds + \frac{dv_1(t)}{dt} \int_{-D/\sqrt{8}}^{L} ds$$

$$= \left( \frac{D}{\sqrt{8}} + L + \frac{D}{\sqrt{8}} \right) \cdot \frac{dv_1(t)}{dt} = \left( \frac{D}{\sqrt{2}} + L \right) \frac{dv_1(t)}{dt}$$

well rounde inlet

if $L \gg D \rightarrow \int_{-\infty}^{L} \frac{\partial v}{\partial t} ds = L \cdot \frac{dv_1(t)}{dt}$
Example: duct from a big tank

$L = 20m \gg D, \ L_1 = 5m$

$h = 5m$

$\rho = 10^3 \frac{kg}{m^3}, \ g = 10\frac{m}{s^2}$

a) At what time after the immediately opening of the flap the flow reaches 99% of its final value?
b) At that time, what is the difference between the current pressure and the final pressure at point “A”?
Example: duct from a big tank

a) Bernoulli: 
\[ p_a + \rho g (h + s) = p_a + \rho g s + \frac{\rho}{2} v_4^2 + \rho \int_{s_0}^{s_4} \frac{\partial v}{\partial t} ds \]

well rounded inlet: 
\[ \int_{s_0}^{s_4} \frac{s_4 \partial v}{\partial t} ds = L \frac{dv_4}{dt} \]

\[ \rightarrow \rho g h = \frac{\rho}{2} v_4^2 + L \rho \frac{dv_4}{dt} \quad \rightarrow \int_0^T dt = L \frac{dv_4}{gh - \frac{v_4^2}{2}} \]

Integration

\[ T = 2L \int_0^{0.99\sqrt{2gh}} \frac{dv_4}{\sqrt{2gh - v_4^2}} \]

\[ = \frac{L}{\sqrt{2gh}} \ln \left[ \frac{\sqrt{2gh + v_4}}{\sqrt{2gh - v_4}} \right]_0^{0.99\sqrt{2gh}} = 10.6 \text{ s} \]
the accelerated initial flow is dependent of \( L \), but \( v_4(t \to \infty) \) is independent of \( L \).

\[ p_a = p_A + \rho g h_1 + \frac{\rho}{2} v_4^2 + L_1 \frac{dv_4}{dt} \]

\[ t \to \infty : p_a = p_{A,\infty} + \rho g h_1 + \frac{\rho}{2} 2gh \]

from a) \[ \frac{dv_4}{dt} = \frac{1}{L} \left( g h - \frac{v_4^2}{2} \right) \]

\[ \Rightarrow p_A - p_{A,\infty} = \rho g h \left( 1 - 0.99^2 \right) \left( 1 - \frac{L_1}{L} \right) = 746 \, \frac{N}{m^2} \]
A piston is moving in a duct: \( s = s_0 \cdot \sin \omega t \)

\[
\begin{align*}
    p_a &= 1 \text{ bar} \quad L = 10 \text{ m} \ll D \quad h = 2 \text{ m} \quad g = 10 \frac{m}{s^2} \\
    s_0 &= 0.1 \text{ m} \quad \rho = 10^3 \frac{kg}{m^3} \quad p_D = 2500 \frac{N}{m^2}
\end{align*}
\]

At what angular speed \( \omega \) the pressure at the bottom of the piston reaches the vapour pressure \( p_D \)?
example: moving piston

\[ p_a = p_P + \rho gh + \frac{\rho v_P^2}{2} + \rho \int_{s_1}^{s_P} \frac{\partial v}{\partial t} ds \]

\[ s_0 \ll L \quad \rightarrow \quad \int_{s_1}^{s_P} \frac{\partial v}{\partial t} ds = L \frac{dv_P}{dt} \]

\[ p_P = p_a - \rho gh + \rho s_0 \omega^2 \left( L \sin \omega t - \frac{s_0}{2} \cos^2 \omega t \right) \]

\[ p_{P,\text{min}} = p_D \]

\[ p_D = p_{P,\text{min at } \cos \omega t = 0} \quad \rightarrow \quad \frac{dp_P}{dt} = 0 \]

\[ \omega = \sqrt{\frac{p_a - p_D - \rho gh}{\rho s_0 L}} = 8.8 \text{ s}^{-1} \]
2 balloons are connected with a pipe of length $L$ and the cross-section $A$. The pressure in the balloons depends linearly on the balloon volume. $V_0$ is the volume at ambient pressure.

\[ p = p_a + C(V - V_0) \]

At $t = 0$ one balloon is compressed by the volume $\Delta V$. 

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**example: balloons**
example: balloons

a) Show, that the frictionless flow in the pipe is described by the equation of oscillation

\[ \ddot{u} + K^2 u = 0. \]

Determine the eigenfrequency of the system.

b) Determine the maximum of the velocity in the pipe.

c) Determine the maximum of the pressure difference between the balloons.

Given: \( \Delta V, L, A, \rho, C \)

Hint:

• General Ansatz for the equation of oscillations

\[ \ddot{x} + a^2 x = 0: \]

\[ x = C_1 \sin(at) + C_2 \cos(at) \]
example: balloons

a)

continuity for the balloons:

\[ \dot{V}_1 = -uA \quad \dot{V}_2 = uA \]

Bernoulli (unsteady, frictionless)

\[ \rho L \ddot{u} + p_2 - p_1 = 0 \]

\[ \Rightarrow \rho L \ddot{u} + \dot{p}_2 - \dot{p}_1 = 0 \]

\[ p = p_a + C(V - V_0) \Rightarrow \dot{p} = C\dot{V} \]

\[ \Rightarrow \rho L \ddot{u} + 2CAu = 0 \Rightarrow \ddot{u} + \frac{2CA}{\rho L}u = 0 \Rightarrow K = \sqrt{\frac{2CA}{\rho L}} \]
example: balloons

b) 

\[ V_2(t) = V + 0 - \Delta V \cos(Kt) \Rightarrow \dot{V}_2(t) = \Delta VK \sin(Kt) = uA \]

\[ \Rightarrow u(t) = \frac{\Delta VK}{A} \sin(Kt) \Rightarrow u_{max} = \frac{\Delta VK}{A} = \Delta V \sqrt{\frac{2C}{\rho AL}} \]

c) 

\[ |p_2 - p_1|_{max} = |\rho L \dot{u}|_{max} \]

\[ \dot{u} = \frac{\Delta VK^2}{A} \cos(Kt) \Rightarrow |p_2 - p_1|_{max} = \frac{\rho L \Delta V}{A} K^2 = 2C \Delta V \]
The flap at the exit of the water pipe (constant width $B$) of a large container is opened abruptly. The appearing flow is without any losses.

Given: $H$, $h_1$, $h_2$, $g$, $L$ ; $L >> h_1$
Determine

a) the differential equation for the exit velocity $v_3$

b) - the local acceleration
  - the convective acceleration
  - the substantial acceleration

  at $x = \frac{L}{2}$ when the exit velocity reaches half of its asymptotic final value!

Hint: The computation of $v(t)$ is not necessary for solving this problem.
Bernoulli from "0" to "3"

\[ p_a + \rho \ g \ H = p_3 + \frac{\rho}{2} v_3^2 + \int_0^3 \rho \left. \frac{\partial v}{\partial t} \right|_{s} ds , \quad p_3 = p_a \]

Splitting of the integral

\[ \int_0^1 \rho \left. \frac{\partial v}{\partial t} \right|_{s} ds \approx 0(h_1 << L) \]

\[ \int_1^2 \rho \left. \frac{\partial v}{\partial t} \right|_{s} ds , \quad v = v_2 \frac{h_2}{h} , \quad h = h_1 + \frac{h_2 - h_1}{L} x \]

\[ \Rightarrow \rho \left. \frac{dv_2}{dt} \right|_{1}^2 \left. \frac{h_2}{h_1 + \frac{h_2 - h_1}{L} x} \right| dx = \rho \left. \frac{dv_2}{dt} \right|_{1}^2 \frac{h_2 L}{h_2 - h_1} \ln \frac{h_2}{h_1} = \rho \left. \frac{dv_2}{dt} \right|_{1}^2 \frac{L}{h_2 - h_1} x \]
\[ \rho \int_2^3 \frac{\partial v}{\partial t} \, ds = \rho L \frac{dv_2}{dt} \]

introduce in Bernoulli

\[ p_a + \rho g H = p_a + \frac{\rho}{2} v_3^2 + \rho \frac{dv_3}{dt} (L + L) \]

\[ \frac{dv_3}{dt} = \frac{1}{L+L} \left( g H - \frac{v_3^2}{2} \right) \]

\[ t \to \infty : g H - \frac{1}{2} v_{3e}^2 = 0 \implies v_{3e} = \sqrt{2 g H} \]

local acceleration:

\[ b_l = \frac{\partial v}{\partial t} = \frac{dv_3}{dt} \frac{h_2}{h}, \quad b_l (v_3=\frac{1}{2} v_{3e}, x=L) = \frac{1}{L+L} g H \frac{3}{4} \frac{2 h_2}{h_1 + h_2} \]
convective acceleration:

\[ v = v_2 \frac{h_2}{h}, \quad \frac{\partial v}{\partial x} = -v_2h_2\frac{1}{h^2} \]

\[ b_k = v \frac{\partial v}{\partial x} = -v_3^2 \frac{h_2^2}{h^3} \frac{dh}{dx}, \quad \frac{dh}{dx} = \frac{h_2 - h_1}{L} \]

\[ b_k(v_3 = \frac{1}{2} v_{3e}, x = \frac{L}{2}) = 4 \frac{g H}{L} \frac{h_2^2 (h_1 - h_2)}{(h_1 + h_2)^3} \]

substantial acceleration:

\[ b_s = b_l + b_k = \frac{3}{2} \frac{g H}{L + L} \frac{h_2}{h_1 + h_2} + 4 \frac{g H}{L} \frac{h_2^2 (h_1 - h_2)}{(h_1 + h_2)^3} \]