Potential theory

Requirements: frictionless, no rotation
2-dimensional (planar)
incompressible, steady

Rotation free: \( \vec{\omega} = 0 \)
\[
\vec{\omega} = \frac{1}{2} \text{ rot } \vec{v} = \frac{1}{2} \vec{a} \times \vec{v} = \frac{1}{2} \begin{pmatrix}
    w_y - v_x \\
    u_z - w_x \\
    v_x - u_y
\end{pmatrix}
\]

2-dimensional flow: \( w_x = w_y = 0 \)

\[
\omega_z = \frac{1}{2} (v_x - u_y) = \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = 0
\]

Rotational flow

[Example 4.1] \( \text{rot} (\text{grad} \psi) = \vec{0} \)
If \( \omega = 0 \) \( \rightarrow \) A function \( \phi \) exists with the property \( \nabla = \nabla \phi \rightarrow (v) = (\partial \phi / \partial x) \)
Potential

\( \rightarrow \) Continuity (2-D, steady, incompressible)

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \nabla \cdot \vec{v} = 0 \implies \nabla^2 \phi = \Delta \phi = 0
\]

\[
\Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0
\]

Linear differential equation

\( \Rightarrow \) The principle of superposition is applicable

If \( \phi_1, \phi_2 \) are solutions of the equation,

then \( c_1 \phi_1, c_2 \phi_2 \) and \( c_1 \phi_1 + c_2 \phi_2 \) are also solutions

Stream function: \( u = \frac{\partial \psi}{\partial y} \); \( v = -\frac{\partial \psi}{\partial x} \) (fulfills the Continuity)

\[ \omega = 0 \implies \nabla^2 \phi = \Delta \phi = 0 \]

\( \phi_x = \psi_y \); \( \phi_y = -\psi_x \rightarrow \phi \) and \( \psi \) are perpendicular
\( \phi = \text{constant} \rightarrow \text{lines of constant potential} \)
\( \psi = \text{constant} \rightarrow \text{streamlines} \)

\( \phi \) and \( \psi \) are used to describe flow fields and flows around bodies.

The contour is provided by a special streamline

\( \Rightarrow \) The velocity vector \( \mathbf{v} \) is parallel the wall

But: Stokes no-slip condition cannot be fulfilled (frictionless + rotation free)

\( \rightarrow \) Drag forces and viscous stresses cannot be determined

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- **Complex numbers**:
  
  \[ z = x + i y = \tau e^{i\psi} = \tau (\cos \psi + i \sin \psi) \]
  
  \[ x = \tau \cos \psi \]
  
  \[ y = \tau \sin \psi \quad \Leftrightarrow \quad \tau = \sqrt{x^2 + y^2} \]
  
  \[ \psi = \arctan \left( \frac{y}{x} \right) \]

- **Complex velocity**
  
  \[ w = u + iv \]

- **Conjugate complex velocity**
  
  \[ \overline{w} = u - iv \]
complex potential function
complex stream function
\[ \mathcal{F}(z) = \int \tilde{w} \, dz = \phi(x,y) + i \psi(x,y) \]
\[ \Rightarrow \text{Laplace equation} \]
\[ \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) + i \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) = 0 \]
\[ \tilde{w} = u - iv = \frac{d\mathcal{F}}{dz} \]

**Singularities:**

**Parallel flow:**
\[ \mathcal{F}(z) = (U_\infty - iV_\infty) \cdot z \]
\[ \phi = U_\infty x + V_\infty y \]
\[ \psi = U_\infty y - V_\infty x \]
\[ u = U_\infty \quad v = V_\infty \]

\[ \text{Stream lines} \]

**Source, sink:**
\[ \mathcal{F}(z) = \frac{E}{2\pi} \ln z \]
\[ \phi = \frac{E}{2\pi} \ln r \]
\[ \psi = \frac{E}{2\pi} \phi \]
\[ u = \frac{E}{2\pi} \frac{x}{x^2 + y^2} \]
\[ v = \frac{E}{2\pi} \frac{y}{x^2 + y^2} \]
Vortex: \[ \tau(z) = \frac{i}{2\pi} \ln \frac{1}{z} \]

\[ \phi = -\frac{\Gamma}{2\pi} \arctan \frac{Y}{x} \quad \zeta = \frac{\pi}{2\pi} \ln \sqrt{x^2 + y^2} \]

\[ u = \frac{\Gamma}{2\pi} \frac{Y}{x^2 + y^2} \quad v = -\frac{\Gamma}{2\pi} \frac{x}{x^2 + y^2} \]

Dipole, Doublet \[ \tau(z) = \frac{u}{z} \]

\[ \phi = \frac{mx}{x^2 + y^2} \quad \zeta = -\frac{uy}{x^2 + y^2} \]

\[ u = um \frac{y^2 - x^2}{(x^2 + y^2)^2} \quad v = -um \frac{2xy}{x^2 y^2} \]
Corner flow

\[ f(z) = \frac{a}{n} z^n \quad n \in \mathbb{R} \]

\[ a \in \mathbb{C} \]

\[ \phi = \frac{a}{n} \cdot r \cdot \cos(n \theta) \quad \theta = \frac{a}{n} \cdot r \cdot \sin(n \theta) \]

\[ \begin{align*}
\text{acute angle} & \quad \text{concave} \\
\text{convex} & \quad \text{concave}
\end{align*} \]

Singalurities have their center in the origin of the coordinate system.

\[ f(z) = \frac{E}{2\pi} \ln(z - a - ib) \]

\[ f(z) = \frac{E}{2\pi} \ln(z - a) \]

Example
• Simulation of walls by mirroring

\[
\begin{align*}
\bullet & \quad u_0 \to \bullet \quad = \quad u_0 \to \bullet
\end{align*}
\]

\[\text{Symmetry plane}\]

• Usually, the contours are preceded by the stagnation streamlines
  • locate the stagnation point \((u = v = 0)\)
  • determine \(\theta\) at the stagnation point
  • sketch the streamlines

\[
\phi_2(x, y) = \phi_1(x_s, y_s) = \text{const.}
\]

• Streamlines do not intersect
  → each streamline can represent a body contour. In this case usually \(u_w \neq 0\)

\[\text{[Sketch of streamlines]}\]

• Bernoulli's equation is valid

\[
\begin{align*}
\rho_0 &= P_0 + \frac{1}{2} \rho \left( u_0^2 + v_0^2 \right) = P + \frac{1}{2} \rho \left( u^2 + v^2 \right) = \text{const.}
\end{align*}
\]

• Compute \(\phi_0 = \frac{P - P_{ref}}{\frac{3}{2} u_{ref}^2} = \frac{\frac{1}{2} \rho u_y^2 - \frac{1}{2} \rho v^2}{\frac{3}{2} u_{ref}^2} = \lambda - \frac{v^2}{u_{ref}^2}\)
14.5 A planar flow is described by the stream function \( \psi = \left( \frac{U}{L} \right) xy \).

In \( x_{ref} = 0, y_{ref} = 1 \) m the pressure is \( p_{ref} = 10^5 \) N/m\(^2\).

\[
U = 2 \text{ m/s} \quad L = 1 \text{ m} \quad \rho = 10^3 \text{ kg/m}^3
\]

a) Proof, if the flow has a potential.

Determine

b) the stagnation points, the pressure coefficient, and the lines of constant total velocity

c) the velocity and the pressure at \( x_1 = 2m, y_1 = 2m, \)

d) the coordinates of a particle at \( t = 0.5s \) if it passes at \( t = 0 \) the point \( x_1, y_1, , \)

e) the pressure difference between these two points.

f) Sketch the streamlines.
a) Given stream function \( \psi = \frac{U}{L} xy \)

\( \phi \) exists if \( \omega = 0 \)

Planar flow \( \to \) 2-dimensional \( \to \) \( \omega_x = \omega_y = 0 \)

\( \omega_2 = \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \)

\[
\begin{align*}
 u &= \frac{\partial \psi}{\partial y} = \frac{U}{L} x \quad \Rightarrow \quad \frac{\partial u}{\partial y} = 0 \\
 v &= -\frac{\partial \psi}{\partial x} = -\frac{U}{L} y \quad \Rightarrow \quad \frac{\partial v}{\partial x} = 0
\end{align*}
\]

\( \omega_2 = 0 \)

\( \Rightarrow \) the flow is rotation free and the potential exists

\( \phi \) exists \( \Rightarrow \) computation of \( \phi \)

1. \( u = \frac{\partial \phi}{\partial x} \to \phi = \int u \, dx + f_1(y) + C_1 \)

2. \( v = \frac{\partial \phi}{\partial y} \to \phi = \int v \, dy + f_2(x) + C_2 \)

1. \( \phi(x, y) = \int \frac{U}{L} x \, dx + f_1(y) + C_1 \)

2. \( \phi(x, y) = \int -\frac{U}{L} y \, dy + f_2(x) + C_2 \)
1. \( \phi(x, y) = \frac{u}{L} \frac{x^2}{2} + f_1(y) + c_1 \)

2. \( \phi(x, y) = -\frac{u}{L} \frac{y^2}{2} + f_2(x) + c_2 \)

**Comparison of 1.) and 2.)**

\[
\frac{u}{L} \frac{x^2}{2} + f_1(y) + c_1 = -\frac{u}{L} \frac{y^2}{2} + f_2(x) + c_2
\]

\[
f_1(y) = -\frac{u}{L} \frac{y^2}{2} \quad \text{and} \quad f_2(x) = \frac{u}{L} \frac{x^2}{2}
\]

\[
c_1 = c_2 = c
\]

\[
\phi = \frac{u}{2L} (x^2 - y^2) + c
\]

**Complex potential \( \mathcal{F}(z) \)**

\[
\mathcal{F}(z) = \mathcal{F}(x + iy) = \phi(x, y) + i \xi(x, y)
\]

\[
= \frac{u}{2L} (x^2 - y^2) + i \frac{u}{2} xy
\]

\[
= \frac{u}{2L} (x^2 + 2ixy - y^2)
\]

\[
= \frac{u}{2L} z^2
\]
Streak of the flow field
- stagnation points - stagnation streamline
- asymptotic streamlines for
  \[ x, y \to \infty ; x, y \to 0 \]
- flow direction

\[ \text{Stagnation point: } \vec{V} = \vec{0} : u = v = 0 \]
\[ u = \frac{U}{L} x, \quad v = -\frac{U}{L} y \to (x_s, y_s) = (0, 0) \]

Additionally: \( u = 0 \) on the \( y \)-axis
\( v = 0 \) on the \( x \)-axis

\[ \text{Stream lines: } \xi = \text{const.} \]
\[ \xi = \frac{U}{L} xy = \text{const.} \]
\[ \rightarrow y = \frac{L}{U} \text{const.} \quad \frac{1}{x} = \frac{C}{x} \quad \text{for } x \neq 0 \]
\[ x = \frac{L}{U} \text{const} \quad \frac{1}{y} = \frac{C}{y} \quad \text{for } y \neq 0 \]
\[ \rightarrow \text{Hyperbola} \]
Stagnation point stream line

\[ y_{sp} = \frac{U}{L} x_{sp} \quad y_{sp} = 0 \quad \text{dependend on the problem} \]

\[ y = 0 \rightarrow x = 0 \quad \text{or} \quad y = 0 \]

\[ \rightarrow x\text{-axis and } y\text{-axis are stagnation stream lines} \]

flow direction

\[ u = \frac{u}{L} x; \quad v = -\frac{u}{L} y \]

\[ u \leq 0 \quad v \leq 0 \]

\[ u \geq 0 \quad v \leq 0 \]

\[ u \leq 0 \quad v \geq 0 \]

\[ u \geq 0 \quad v \geq 0 \]
Pressure coefficient

\[ c_p = \frac{P - P_{inf}}{\frac{1}{2} \rho V_{inf}^2} = \lambda - \frac{\frac{\rho}{2} V^2}{V_{inf}^2} = \lambda - \frac{U^2 + V^2}{U_{inf}^2 + V_{inf}^2} \]

\[ u = \frac{U}{L} \]
\[ v = -\frac{U}{L} \]

Lines of constant velocity

\[ |\mathbf{V}| = \sqrt{U^2 + V^2} = \text{constant} \]

\[ = \sqrt{\left(\frac{U}{L} x\right)^2 + \left(-\frac{U}{L} y\right)^2} = \frac{L}{U} \sqrt{U^2} = x^2 + y^2 \]

circle with the radius \( R = \frac{|\mathbf{V}| L}{U} \)
$90^\circ$ - Corner flow

Planar stagnation point flow
14.4 The complex stream function is given

\[ F(z) = \frac{2}{3} \frac{u_{\infty}}{\sqrt{L}} z^{\frac{3}{2}} + \frac{E}{2\pi} \ln(z) \]

Given: \( L, \ u_{\infty} \)

Determine

a) the potential \( \phi(r, \theta) \) and the stream function \( \psi(r, \theta) \).
b) the components of the velocity \( v_r, v_\theta \).
c) the constant \( E \) such that a stagnation point is at \( (x = -L, y = 0) \).
d) the equation that describes the contour \( r_k(\theta) \).

Hints:

\[
\begin{align*}
z &= x' + iy' = re^{i\theta} \\
v_r &= \frac{\partial \phi}{\partial r} = \frac{1}{r} \frac{\partial \psi}{\partial \theta}, \quad v_\theta = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}
\end{align*}
\]
\( a) \ \bar{V}(t) = \frac{2}{3} \frac{u_{oo}}{V_L} t^{2/3} + \frac{E}{2\pi} \ln t \)

\[ = \frac{2}{3} \frac{u_{oo}}{V_L} \sqrt[3]{t} e^{-i \frac{3}{2} \theta} + \frac{E}{2\pi} (k \cdot \tau + k \cdot e^{-i \theta}) \]

\[ \phi(t, \theta) = \Phi(t(t)) = \frac{2}{3} \frac{u_{oo}}{V_L} t^{3/2} \cos \left( \frac{3}{2} \theta \right) + \frac{E}{2\pi} \ln t \]

\[ \psi(t, \theta) = \Im(t(t)) = \frac{2}{3} \frac{u_{oo}}{V_L} t^{3/2} \sin \left( \frac{3}{2} \theta \right) + \frac{E}{2\pi} \theta \]

\[ b) \ \bar{V}(r, \theta) = \frac{\partial \phi}{\partial r} = \frac{u_{oo}}{V_L} \sqrt{\frac{r}{L}} \cos \left( \frac{3}{2} \theta \right) + \frac{E}{2\pi} \]

\[ \bar{V}_{\theta}(r, \theta) = \frac{1}{r} \frac{\partial \phi}{\partial \theta} = - \frac{u_{oo}}{V_L} \sqrt{\frac{r}{L}} \sin \left( \frac{3}{2} \theta \right) \]

\[ c) \ \text{ stagnation point at } (x = -L, \ y = 0) \rightarrow (r = 2, \ \theta = \frac{2}{3} \pi) \]

\[ \bar{V}_{\theta} = - \frac{u_{oo}}{V_L} \sqrt{\frac{L}{L}} \sin \frac{\pi}{3} = 0 \]

\[ \bar{V}_{r} = \frac{u_{oo}}{V_L} \sqrt{\frac{L}{L}} \cos \frac{\pi}{3} + \frac{E}{2\pi} L = 0 \rightarrow E \approx 2 \frac{\pi}{L} \]

\[ d) \ \bar{v}_{sp} = \frac{E}{2\pi} \cdot \frac{2}{3} \bar{u} = \frac{E}{3} = \frac{2}{3} \frac{\pi}{L} u_{oo} L = 2 \bar{u} \]

\[ \frac{2}{3} \bar{u} u_{oo} L = \frac{2}{3} \frac{u_{oo}}{V_L} t^{3/2} \sin \left( \frac{3}{2} \theta \right) + u_{oo} L \theta \]

\[ \rightarrow \bar{v}_{\theta}(\theta) = L \cdot \left( \frac{\bar{u} - \frac{2}{3} \theta}{\sin \frac{3}{2} \theta} \right)^{1/2} \]