Turbulent flows

Reynolds averaging: splitting of the turbulent velocity $\vec{v}$ in an average value $\vec{\bar{v}}$ and a fluctuation $\vec{v}'$

$$\vec{v} = \vec{\bar{v}} + \vec{v}'$$

Example: Pipe

fully turbulent symmetric flow
Turbulent flows

\[
\begin{align*}
  u(r, \phi, x, t) &= \overline{u}(r) + u'(r, \phi, x, t) \\
  v(r, \phi, x, t) &= v'(r, \phi, x, t)
\end{align*}
\]

definition:

\[
\overline{u} = \frac{1}{T} \int_T u(x, y, z, t) \, dt
\]

\[
\rightarrow \overline{u} = \overline{u}(x, y, z) \neq f(t) \quad u' = u - \overline{u}
\]
Computational rules

\[ \overline{f'} = 0 \quad \text{average of the fluctuation} \]

\[ \overline{f} = \overline{\overline{f}} \quad \text{average of the average} \]

\[ \text{konst} \neq f(t) \]

\[ \overline{f + g} = \frac{1}{T} \int_{T} (f + g) dt = \frac{1}{T} \int_{T} f dt + \frac{1}{T} \int_{T} g dt = \overline{f} + \overline{g} \]

\[ \overline{fg} = \overline{f} \overline{g} : \quad g \neq g(t) \rightarrow \frac{1}{T} \int_{T} fg dt = \frac{1}{T} \overline{g} \int_{T} f dt = \overline{f} \overline{g} \]

\[ \overline{\frac{\partial f}{\partial x}} = \overline{\frac{\partial f}{\partial x}} \quad \text{average of the derivative} \]
Computational rules

\[
\overline{fg} = \frac{1}{T} \int_{T} fg \, dt = \frac{1}{T} \int_{T} (\overline{f} + f')(\overline{g} + g') \, dt
\]

\[
= \frac{1}{T} \int_{T} (\overline{fg} + f'\overline{g} + \overline{fg}' + f'g') \, dt
\]

\[
= \overline{fg} + \overline{fg}' \quad \text{usually} \neq 0, \text{ z. B. } f = g \rightarrow \overline{f^2} \neq 0
\]

level of turbulence

\[
Tu = \frac{1}{u_{\infty}} \sqrt{\frac{1}{3} u'^{2} + v'^{2} + w'^{2}}
\]

turbulent intensity
3-D, incompressible, unsteady momentum equation

konvective term: $\frac{\partial v_k v_j}{\partial x_k}$

z. B.: $\frac{\partial u v}{\partial x}$; $\frac{\partial u w}{\partial y}$

for turbulent flows: average of the complete equation

$$\rightarrow \frac{\partial v_k v_j}{\partial x_k} = \frac{\partial}{\partial x_k} = (v_k v_j + v'_k v'_j)$$

additional term

$$-\rho v'_k v'_j$$

turbulent shear stress tensor
Bernoulli equation (Energy equation) for pipe flows with loss of the total pressure

\[ p_{01} = p_{02} + \Delta p_v \]

Total pressure loss
\[ p_1 + \frac{\rho}{2}u_{m1}^2 + \rho g z_1 = p_2 + \frac{\rho}{2}u_{m2}^2 + \rho g z_2 + \Delta p_v \]

\[ \Delta p_v = \sum (\xi_i + \lambda_i \frac{L_i}{D_i}) \frac{\rho}{2}u_{m_i}^2 \]

\( \xi_i \triangleq \) pressure loss coefficient for special places, where losses occur (inlet, unsteady enlargement of cross section, ellbow, . . .)

\( \lambda_i \triangleq \) loss coefficient in straight pipes

\( u_{m_i} \triangleq \) average velocity
examples for pressure loss coefficients

usually: determine $\zeta$ from experiments

$$\xi = \xi(\text{Re}, \text{geometry})$$
unsteady enlargement of cross section

\[ \zeta_E = \frac{\Delta p}{\frac{1}{2} \rho u_{m1}^2} = (1 - \frac{A_1}{A_2})^2 \]

Carnot caution

laminar flow, inlet, circular pipes

\[ \rightarrow 1.12 \leq \zeta_e \leq 1.45 \] experimental
pressure loss coefficients for pipes (smooth pipes)

\[ \text{Re} = \frac{\bar{u} \rho D}{\eta} \]

- **laminar:** \( (\text{Re} \leq 2.300) \)
  \[ \lambda = \frac{C}{\text{Re}} \]
  \[ C = 64 \] for circular cross-sections (Hagen-Poiseuille)

- **turbulent:** Blasius \( (2.300 \leq \text{Re} \leq 10^5) \)
  \[ \lambda = \frac{0.316}{\sqrt[4]{\text{Re}}} \]

**iterative solution:** Prandtl:
\[ \frac{1}{\sqrt{\lambda}} = 2 \log(\text{Re} \sqrt{\lambda}) - 0.8 \]
**Reference velocity**

viscous effects in the pipe

\[ \Delta p = \lambda \frac{L \rho u^2_m}{D^2} \]

average pipe velocity

\[ \Delta p_v = \xi_e \frac{\rho u^2_m}{2} \]

average pipe velocity
Reference velocity

unsteady change of cross section

\[ \Delta p_v = \xi E \frac{\rho u_{m,1}^2}{2} \]

incoming velocity
typical problem (losses)

\[ h = 10 \text{ m} \quad D = 0.05 \text{ m} \quad L = 4 \text{ m} \quad \beta = 0.25 \]

\[ \alpha = 4.4, \gamma = 0.025 \]

typical problem (losses)

Bemerkung:

- mechanical losses are known \((\zeta, \lambda)\)
- the flow in the inlet and in the nozzle is lossfree
- the flow in the pipes is fully developed
typical problem (losses)

→ Bernoulli

\[ p_0^1 = p_0^2 + \Delta p_v \]

- total available energy
- still existing energy
- total energy in ‘2’
- transformed energy
- → inner energy
- total pressure loss

\[ p_0 = p + \frac{\rho}{2} u^2 + \rho g z \]

\[ \Delta p_v = \sum (\xi_i + \lambda_i \frac{L_i}{D_i}) \frac{\rho}{2} u_{m_i}^2 \]
Bernoulli from 'd' → 'H' ($u_H = 0$)

$$p_a + \frac{\rho}{2} u_d^2 = p_a + \rho g H$$

$$\rightarrow H = \frac{u_d^2}{2g} \rightarrow \text{unknown} \quad u_d ?$$

extended Bernoulli

$$p_{01} = p_a + \rho gh = p_a + \frac{\rho}{2} u_d^2 + \frac{\rho}{2} u_{mD}^2 \left( 2\xi_K + \xi_v + \frac{L}{D} \right)$$

nozzle velocity  pipe velocity

continuity:  \( u_{mD} A_D = u_d A_d \rightarrow u_{mD} = u_d \left( \frac{d}{D} \right)^2 \)
### Typical Problem (Losses)

<table>
<thead>
<tr>
<th>Lossfree</th>
<th>With Losses</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \xi_K = \xi_v = \lambda = 0 )</td>
<td>( \rho gh = \frac{\rho}{2} u_d^2 + \rho 2 u_d^2 \left( \frac{d}{D} \right)^4 K )</td>
</tr>
<tr>
<td>( u_d = \sqrt{2gh} )</td>
<td>( u_d = \sqrt{\frac{2gh}{1 + \left( \frac{d}{D} \right)^4 K}} )</td>
</tr>
</tbody>
</table>

Volume flux: \( \dot{Q} = \frac{\pi}{4} d^2 u_d \)

\[
\dot{Q} = \frac{\pi}{4} \sqrt{2ghD^2} \frac{d^2}{D^2} \left( \frac{d}{D} \right)^2
\]

\( \dot{Q} \sim \left( \frac{d}{D} \right)^2 \)
typical problem (losses)

\[ \dot{Q} \]

\[ \frac{d^2}{D^2} \]

verlustfrei

verlustbehaftet
typical problem (losses)

ceiling of the fountain

\[ H = \frac{u_d^2}{2g} \]

<table>
<thead>
<tr>
<th>no losses</th>
<th>with losses</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H = h )</td>
<td>( H = \frac{h}{1 + (\frac{d}{D})^4 K} )</td>
</tr>
</tbody>
</table>

influence of \( \frac{d}{D} \)

\( \frac{d}{D} \downarrow \rightarrow H \uparrow \)
The velocity profile in a fully developed flow in a pipe with a smooth surface can be approximated with the potential law:

\[
\frac{\bar{v}}{\bar{v}_{max}} = \left(1 - \frac{r}{R}\right)^{\frac{1}{n}}, \text{ mit } n = n(Re).
\]

<table>
<thead>
<tr>
<th>$Re$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \cdot 10^5$</td>
<td>7</td>
</tr>
<tr>
<td>$6 \cdot 10^5$</td>
<td>8</td>
</tr>
<tr>
<td>$1.2 \cdot 10^6$</td>
<td>9</td>
</tr>
<tr>
<td>$2 \cdot 10^6$</td>
<td>10</td>
</tr>
</tbody>
</table>
a) Use the continuity equation to compute the relation between the average velocity $\bar{v}_m$ and the maximum velocity $\bar{v}_{max}$, i.e. $\frac{\bar{v}_m}{\bar{v}_{max}} = f(n)$.

b) At what position $\frac{r}{R}$ is $\bar{v}(r/R) = \bar{v}_m$?

c) How can the results of a) and b) be used, if the volume flux shall be measured?
The ratio between the average and the maximum velocity is

\[
\frac{\bar{v}_m}{\bar{v}_{max}} = 2 \int_0^1 \xi (1 - \xi)^\frac{1}{n} \, d\xi = \frac{2n^2}{(n + 1)(2n + 1)} \quad \text{mit} \quad \xi = \frac{r}{R} .
\]

The integral is solved using partial integration. The average velocity is at a distance

\[
\frac{r_m}{R} = 1 - \left( \frac{\bar{v}_m}{\bar{v}_{max}} \right)^n
\]

see table.
Measuring $\bar{v}(r)$ at a distance $R - r_m$ from the wall, and with the known $\bar{v}_{max}$ the average velocity can be determined, and the volume flux $\dot{V} = v_m \pi R^2$ can be computed.
The pressure decrease $\Delta p$ along $L$ is measured in a fully developed pipe flow with the volume flux $\dot{V}$.

\[
\dot{V} = 0.393 \, m^3/s \quad L = 100 \, m \quad D = 0.5 \, m \quad \Delta p = 12820 \, N/m^2 \quad \rho = 900 \, kg/m^3 \quad \eta = 5 \cdot 10^{-3} \, Ns/m^2
\]
Determine

a) the skin-friction coefficient,
b) the equivalent roughness of the pipe,
c) the wall shear stress and the force of the support.
d) What is the pressure decrease, if the pipe is smooth?
\( \lambda \)

- laminar
- Transition
- turbulent technisch rauh
- turbulent vollkommen rauh
- glatt

\[ \text{Re} = \frac{\bar{u} D}{v} \]
a)

\[ \Delta p = \lambda \frac{L}{D} \frac{\rho}{2} \bar{u}_m^2 \]

\[ \dot{V} = \bar{u}_m \frac{\pi D^2}{4} \]

\[ \Rightarrow \lambda = \frac{\pi^2 \Delta p D^5}{8 \rho L \dot{V}^2} = 0.0356 \]
b) \[ Re = \frac{\rho \bar{u}_m D}{\eta} = 1.8 \cdot 10^5 \]

\[ \frac{k_s}{D} = 0.0083 \quad \text{(from Moody diagram)} \]

\[ \Rightarrow \quad k_s = 4.2 \, \text{mm} \]

c) momentum equation for the inner control surface:

\[ \Delta p \frac{\pi D^2}{4} - \tau_W \pi D L = 0 \]

\[ \Rightarrow \quad \tau_W = \Delta p \frac{D}{4L} = 16 \, N/m^2 \]
momentum equation for the outer control surface:

\[ F = -\Delta p \frac{\pi D^2}{4} = -2517 \, N \]

d) \[ \lambda = 0.016 \quad \text{(from diagram)} \]

\[ \Rightarrow \Delta p = 5.8 \cdot 10^3 \, N/m^2 \]