Problem 1  (8 Points)

A two-way turbine is to be used in a wave power plant. The power $P$ of the turbine is independent of the flow direction and proportional to the volume flux $\dot{V}$. The wave motion changes the vertical deflection $\Delta h$ of the inner floating element and displaces or sucks in air through the pipe and the turbine.

The position of the floating element is given by

$$\Delta h(t) = h_{\text{max}} \cdot \sin(\omega t)$$

a) Determine the power $P(\omega, t)$ of the turbine as a function of the angular speed $\omega$ and the time $t$ when air is displaced. Assume the oscillating internal pressure $p_i(\omega, t)$ to be given.

b) Sketch the distribution of the total and the static pressure along a streamline between the points $i$ and $a$ during the zero-crossing of the floating element where $\sin(\omega t) = 0$ and $\cos(\omega t) = 1$.

Given:

$p_a$, $\rho_L$, $A_R$, $A$ with $A \gg A_R$, $L$, $h_{\text{max}}$, $p_i(\omega, t)$, $g$

Hint:

- The flow field can be computed without any losses.
- The vertical dimension of the floating element and the turbine are to be neglected.
- The acceleration in the inlets can be neglected.
- Check the measurement units and signs of your results!
Problem 1

a) Power: \( P = \dot{V} \Delta p_{0,T} = \dot{V} \Delta p_T \Rightarrow \Delta p_T = \frac{P}{\dot{V}} = \frac{P}{Av_i} \)

Bernoulli \( \mathbf{i} \rightarrow \mathbf{a} \):
\[
p_i(\omega, t) + \frac{\rho_L}{2} v_i^2 = p_a + \rho_L g \cdot 2L + \frac{\rho_L}{2} v_R^2 + \Delta p_T + \rho_L \int_i^a \frac{\partial v_R}{\partial t} \, ds
\]

Continuity: \( v_i A = v_R A_R \Rightarrow v_R = \frac{v_i A}{A_R} \)

Displacement (A.S.): \( h(t) = h_i(t) = h_{\text{max}} \cdot \sin(\omega t) \)

Velocity: \( \frac{dh_i}{dt} = v_i(t) = h_{\text{max}} \omega \cdot \cos(\omega t) \)

Acceleration: \( \frac{dv_i}{dt} = -h_{\text{max}} \omega^2 \cdot \sin(\omega t) \)

\[ p_i(\omega, t) + \frac{\rho_L}{2} v_i^2 = p_a + \rho_L g \cdot 2L + \frac{\rho_L}{2} v_i^2 + \frac{P}{Av_i} + \rho_L 2L \cdot \frac{dv_R}{dt} \]

\[ \Rightarrow \frac{P}{Av_i} = p_i(\omega, t) - p_a - \rho_L g \cdot 2L + \frac{\rho_L}{2} v_i^2 \left( 1 - \left( \frac{A}{A_R} \right)^2 \right) - \rho_L 2L \cdot \frac{dv_R}{dt} \]

\[ \Rightarrow P = Av_i \left( p_i(\omega, t) - p_a - \rho_L g \cdot 2L + \frac{\rho_L}{2} v_i^2 \left( 1 - \left( \frac{A}{A_R} \right)^2 \right) - \rho_L 2L \cdot \frac{dv_R}{dt} \right) \]

\[ P = Av_i \left( p_i(\omega, t) - p_a - \rho_L g \cdot 2L + \frac{\rho_L}{2} \left( h_{\text{max}} \omega \cdot \cos(\omega t) \right)^2 \left( 1 - \left( \frac{A}{A_R} \right)^2 \right) \right. \]

\[ + \left. \rho_L 2L \cdot h_{\text{max}} \omega^2 \cdot \sin(\omega t) \right) \]

\[ = \frac{m^2}{m^2} \frac{N}{s} \cdot \frac{N m}{s} = [W] \]

b) Pressure distribution along a streamline \( \mathbf{i} \rightarrow \mathbf{a} \)
Problem 2  (12 Points)

Measurements are performed at a stationary jet engine test facility. That is, the jet engine is fixed and the air velocity distribution is measured at the engine exit and further downstream in the cross section (*).

![Diagram of jet engine test facility]

The velocity distribution $u_1(r)$ at the engine exit is given by:

$$u_1(r) = u_{max_1} \left( 1 - \frac{r^2}{R^2} \right), \quad 0 \leq r \leq R$$

a) Determine the power of the engine.

b) In a certain distance of the engine exit, i.e., in the cross section (*) a velocity profile develops, that is given by:

$$u_2(r) = u_{max_2} \left( 1 - \frac{r^2}{4R_2^2} \right), \quad 0 \leq r \leq R_2$$

Determine the radius $R_2$.

Given:

$R$, $\rho$, $u_{max_1}$

Hint:

- To determine the power the average velocity $u_m$ at the engine exit is to be used.
- The mixing of the jet and the ambient air is to be neglected!
- Check the measurement units and signs of your results!
**Problem 2**

a) Momentum in the x-direction:

\[ P = F_H \cdot u_m \]

\[ \sum \vec{F}_i = \int \vec{u}(\vec{v} \cdot \vec{n})dA \]

\[ F_H = 2\pi \rho \int_0^R u_1(r)^2 r dr \]

\[ = 2\pi \rho u_{max1}^2 \int_0^R \left( 1 - \frac{r^2}{R^2} + \frac{r^4}{R^4} \right) r dr \]

\[ F_H = 2\pi \rho u_{max1}^2 \left( \frac{R^2}{2} - \frac{R^4}{2R^2} + \frac{R^6}{6R^4} \right) \]

\[ = \pi \rho u_{max1}^2 \frac{R^2}{3} \]

\[ u_m = \frac{2}{R^2} \int_0^R u_{max1} \left( 1 - \frac{r^2}{R^2} \right) r dr = 2 \frac{u_{max1}}{R^2} \left( \frac{R^2}{2} - \frac{R^2}{4} \right) = \frac{u_{max1}}{2} \]

\[ \Rightarrow P = \pi \rho u_{max1}^3 \frac{R^2}{6} \text{ [W]} \]

b) continuity:

\[ 2\pi \int_0^R u_{max1} \left( 1 - \frac{r^2}{R^2} \right) r dr = 2\pi \int_0^{R_2} u_{max2} \left( 1 - \frac{r^2}{4R_2^2} \right) r dr \]

\[ u_{max1} \left( \frac{R^2}{2} - \frac{R^2}{4} \right) = u_{max2} \int_0^{R_2} \left( r - \frac{r^3}{4R_2^2} \right) dr \]

\[ u_{max1} \frac{R^2}{4} = u_{max2} \left[ \frac{r^2}{2} - \frac{r^4}{16R_2^2} \right]_0^{R_2} = \frac{7}{16} R_2^2 u_{max2} \]

\[ u_{max1} = \frac{7}{4} \frac{R_2^2}{R^2} u_{max2} \]

momentum in the x-direction:
\[2\pi \rho \int_0^R u_{\text{max}_1}^2 \left(1 - \frac{r^2}{R^2}\right)^2 r \, dr = 2\pi \rho \int_0^{R_2} u_{\text{max}_2}^2 \left(1 - \frac{r^2}{4R_2^2}\right)^2 r \, dr\]

\[u_{\text{max}_1}^2 \left(\frac{R^2}{2} - \frac{R^2}{2} + \frac{R^2}{6}\right) = u_{\text{max}_2}^2 \int_0^{R_2} \left(r - \frac{r^3}{2R_2^2} + \frac{r^5}{16R_2^4}\right) \, dr\]

\[u_{\text{max}_1}^2 \left(\frac{R^2}{6} \right) = u_{\text{max}_2}^2 \left(\frac{R_2^2}{2} - \frac{R_3^2}{8} + \frac{R_4^2}{96}\right)\]

\[\frac{R^2}{6} u_{\text{max}_1}^2 = \frac{37}{96} R_2^2 u_{\text{max}_2}^2\]

\[u_{\text{max}_1} = \frac{\sqrt{37}}{4} \frac{R_2}{R} u_{\text{max}_2}\]

to be plugged into the continuity:

\[\frac{\sqrt{37}}{4} \frac{R_2}{R} u_{\text{max}_2} = \frac{7R_2^2}{4R^2} u_{\text{max}_2}\]

\[\Rightarrow R_2 = \frac{\sqrt{37}}{7} R \quad [m]\]
Problem 3  (9 Points)

The fully developed flow of a Newtonian fluid between coaxial cylinders is considered.

a) Derive the differential equation to determine the shear stress $\tau(r)$ and the velocity $u(r)$ distributions.

b) Determine the velocity $u_a = u(r = a)$ in the $x$-direction of the inner cylinder, if the force onto the inner cylinder vanishes.

Given:
$R, \ a, \ \eta, \ \frac{dp}{dx}$

Hint:
- Check the measurement units and signs of your results!
Problem 3

a) $\tau = \tau(r) \quad \frac{\partial p}{\partial y} = 0 \Rightarrow \frac{\partial p}{\partial x} = \frac{dp}{dx}$

Balance of forces $\sum F_x = 0$:

$$p \cdot 2\pi r dr - \left( p + \frac{dp}{dx} \right) 2\pi r dr + \tau 2\pi r dx - \left( \tau + \frac{d\tau}{dr} dr \right) \left( 2\pi(r + dr) dx - \frac{\tau}{r} \right) dr dr = 0$$

$$\frac{dp}{dx} dr + \frac{d\tau}{r} dr + \tau dr^2 = 0$$

Neglecting higher-order terms:

$$\frac{dp}{dx} + \frac{d\tau}{r} dr + \tau r = 0 \quad \Rightarrow \quad \frac{dp}{dx} + \frac{1}{r} \frac{d\tau}{dr} = 0$$

Newtonian fluid: $\tau = -\eta \frac{du}{dr}$

$$\eta \frac{d}{dr} \left( \frac{du}{dr} \right) - \frac{dp}{dx} = 0$$

b) Boundary conditions:

$$F_R|_{r=a} = 0 \iff \tau|_{r=a} = 0 \iff \frac{du}{dr}|_{r=a} = 0 \quad (1)$$

$$u|_{r=R} = 0 \quad (2)$$

from a):

$$\frac{dp}{dx} dr + \frac{d\tau}{r} dr + \tau r = 0 \quad \Rightarrow \quad \frac{dp}{dx} + \frac{1}{r} \frac{d\tau}{dr} = 0$$

$$\frac{dp}{dx} \cdot \frac{r^2}{2} = \eta \cdot r \cdot \frac{du}{dr} + C_1 \quad \Rightarrow \quad \frac{dp}{dx} \cdot \frac{r^2}{2} = \eta \cdot \frac{du}{dr} + C_1 \cdot \frac{r}{r}$$

$$\eta \frac{d}{dr} \left( \frac{du}{dr} \right) = \frac{dp}{dx}$$

$$\frac{dp}{dx} \cdot \frac{r^2}{4} = \eta \cdot u(r) + C_1 \cdot \ln(r) + C_2$$

with (1):

$$\frac{dp}{dx} \cdot \frac{a^2}{2} = C_1$$

with (2):

$$\frac{dp}{dx} \cdot \frac{R^2}{4} - C_1 \cdot \ln(R) = C_2$$

$$u(r) = \frac{1}{\eta} \frac{dp}{dx} \left( \frac{r^2 - R^2}{4} + \frac{a^2}{2} \ln \left( \frac{R}{r} \right) \right)$$

$$u_a = u|_{r=a} = \frac{1}{\eta} \frac{dp}{dx} \left( \frac{a^2 - R^2}{4} + \frac{a^2}{2} \ln \frac{R}{a} \right) \quad \left[ \frac{s}{kg m} \frac{kg m}{s^2} \frac{1}{m^2} \right] = \left[ \frac{m}{s} \right]$$
The flow \((\rho, \eta, c_p, \lambda, \gamma)\) between two concentric cylinders at a mean axial velocity \(u_\infty\) is considered. The inner cylinder has the diameter \(d\) and the constant wall temperature \(T_0\). The outer cylinder has the diameter \(D\) and the constant wall temperature \(T_W\). Along the length \(L\) the pressure difference \(\Delta p\) is measured.

Among other equations, the flow is described by the radial momentum equation and the energy equation in polar coordinates:

\[
\rho v_r \frac{\partial v_r}{\partial r} = -\frac{\partial p}{\partial r} + \eta \left( \frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} \right) - \rho v_r \frac{\partial e}{\partial r} = \eta \Phi + \lambda \left( \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right).
\]

a) Use the hints and determine the dimensionless parameters of the problem by the method of differential equations.

b) Express the resulting parameters by standard parameters of fluid mechanics.

c) Explain the physical interpretation of the Prandtl number and the Euler number.

Given: all necessary reference values

Hints for a):

- Dimensionless values of the dissipation function \(\Phi\) and the specific internal energy \(e\):

\[
\Phi = \frac{\Phi D^2}{u_\infty^2}, \quad e = \frac{e}{c_p T_0}.
\]

- The quantities density \(\rho\), viscosity \(\eta\), and heat conductivity depend on the temperature and are not constant.
Problem 4

a) dimensionsless quantities: \( \bar{\rho} = \frac{\rho}{\rho_0}, \quad \bar{v_r} = \frac{v_r}{u_\infty}, \quad \bar{p} = \frac{p}{\Delta p}, \quad \bar{T} = \frac{T}{T_0}, \quad \bar{\tau} = \frac{\tau}{c_p T_0}, \quad \bar{\Phi} = \frac{\Phi}{D^2 u_\infty^2} \)

\[ r \text{-momentum:} \quad \rho_0 \frac{u_\infty^2}{D} \rho_0 \partial \bar{v}_r \partial \bar{\tau} = -\frac{\Delta p}{\rho_0 u_\infty^2} \rho_0 \partial \bar{v}_r \partial \bar{\tau} + \eta_0 \frac{u_\infty^2}{D^2} \eta \partial^2 \bar{v}_r \partial \bar{\tau}^2 + \eta_0 \frac{u_\infty}{D^2} \eta \bar{v}_r \bar{\tau} - \eta_0 \frac{u_\infty^2}{D^2} \eta \bar{v}_r \bar{\tau}^2 \]

Division by \( \rho_0 \frac{u_\infty^2}{D} \): \[
\frac{\rho_0}{\rho_0 u_\infty^2} \frac{u_\infty^2}{D} \rho_0 \partial \bar{v}_r \partial \bar{\tau} = -\frac{\Delta p}{\rho_0 u_\infty^2} \rho_0 \partial \bar{v}_r \partial \bar{\tau} + \eta_0 \frac{u_\infty^2}{D^2} \eta \left( \partial^2 \bar{v}_r \partial \bar{\tau}^2 + \frac{1}{\bar{\tau}} \partial \bar{v}_r \partial \bar{\tau} - \frac{\bar{v}_r \bar{\tau}}{\bar{\tau}^2} \right)
\]

\( \Rightarrow \quad K_1 = \frac{\Delta p}{\rho_0 u_\infty^2}, \quad K_2 = \frac{\eta_0}{\rho_0 u_\infty D} \)

Energy: \( \rho_0 \frac{u_\infty^2}{D} c_p T_0 \rho_0 \partial \tau \partial \bar{\tau} = \eta_0 \frac{u_\infty^2}{D^2} \eta \bar{\Phi} + \lambda_0 \frac{T_0}{D^2} \lambda \frac{\partial^2 \bar{T}}{\partial \bar{\tau}^2} + \lambda_0 \frac{T_0}{D^2} \lambda \frac{\partial \bar{T}}{\partial \bar{\tau}} \)

Division by \( \rho_0 \frac{u_\infty^2}{D} c_p T_0 \):

\[ \frac{\rho_0}{\rho_0 u_\infty^2} \frac{u_\infty^2}{D} c_p T_0 \frac{\rho_0}{\rho_0 u_\infty D} \partial \tau \partial \bar{\tau} = \eta_0 \frac{u_\infty^2}{D^2} \eta \frac{\bar{\Phi}}{\bar{T}} + \lambda_0 \frac{T_0}{D^2} \lambda \left( \frac{\partial^2 \bar{T}}{\partial \bar{\tau}^2} + \frac{1}{\bar{T}} \frac{\partial \bar{T}}{\partial \bar{\tau}} \right) \]

\( \Rightarrow \quad K_3 = \frac{\eta_0 u_\infty}{\rho_0 c_p T_0 D}, \quad K_4 = \frac{\lambda_0}{\rho_0 c_p u_\infty D} \)

\( \Rightarrow \) 4 dimensionless parameters.

b) \( K_1 = E_u \)

\[ K_2 = \frac{1}{Re} \]

\[ K_3 = \frac{M^2}{Re} (\gamma - 1) \]

\[ K_4 = \frac{1}{Pr Re} \]

c) Prandtl number: \( \frac{\text{viscous heat}}{\text{conducted heat}} \)

Euler number: \( \frac{\text{pressure force}}{\text{inertia}} \)
Problem 5  (12 Points)

The flow at a constant velocity $u_\infty$ over a flat plate is considered. The volume flux $\dot{V}$ scaled by the width $B$ of the plate $\dot{V}/B$ is extracted through a small slot. The flow shall be investigated using the potential theory.

![Diagram of flow field](image)

a) Determine the complex potential function to describe the flow field.
b) Determine the location of the stagnation point in Cartesian coordinates $(x_s, y_s)$ and the resulting stagnation point streamline in polar coordinates $(r = f(\varphi))$.
c) Determine the minimum height $h$ above the plate for an air particle that is necessary for not being extracted through the slot.
d) Sketch carefully the complete flow field above the plate by specifying the stagnation points and the stagnation point streamlines.

Given:

$u_\infty$, $\dot{V}/B$

Elementary functions:

- parallel flow: $F(z) = (u_\infty - i v_\infty) z$
- potential vortex: $F(z) = \frac{i \Gamma}{2\pi} \ln z$
- source/sink: $F(z) = \frac{E}{2\pi} \ln z$
- stagnation point flow: $F(z) = a z^2$
- dipole: $F(z) = \frac{M}{2\pi z}$
Problem 5

a) Parallel flow + Sink

\[ F(z) = u_\infty z + \frac{E}{2\pi} \ln z \]

b) \[ F(z) = u_\infty (x + iy) + \frac{E}{2\pi} \ln(re^{i\phi}) \]

\[ \rightarrow \phi = u_\infty x + \frac{E}{2\pi} \ln r, \quad r = \sqrt{x^2 + y^2} \]

\[ u = \frac{\partial \phi}{\partial x} = u_\infty + \frac{Ex}{2\pi(x^2 + y^2)}, \quad v = \frac{\partial \phi}{\partial y} = \frac{Ey}{2\pi(x^2 + y^2)} \]

Stagnation point: \( u = v = 0; \quad v = 0 \rightarrow y_s = 0, \quad u = 0 \rightarrow x_s = -\frac{E}{2\pi u_\infty} \)

with \( E = -\frac{2\dot{V}}{B} \rightarrow x_s = \frac{\dot{V}}{Bu_\infty}, \quad y_s = 0 \)

Stagnation point streamline:

\[ \Psi = u_\infty y + \frac{E}{2\pi} \varphi; \quad \varphi_s = \arctan(y_s/x_s) = 0, \quad y_s = 0 \Rightarrow \Psi_s = 0 \]

\[ 0 = u_\infty r \sin \varphi + \frac{E}{2\pi} \varphi \Rightarrow r = \frac{\dot{V}}{Bu_\infty} \cdot \frac{\varphi}{\sin \varphi} \]

c) Volume flux balance: \( \dot{V} < hBu_\infty \Rightarrow h > \frac{\dot{V}}{Bu_\infty} \)

or \( h = \lim_{\varphi \to \pi} y = \lim_{\varphi \to \pi} r \sin \varphi = \lim_{\varphi \to \pi} \frac{\dot{V}}{Bu_\infty} \varphi = \frac{\dot{V}}{Bu_\infty} \)

d) Sketch:
The velocity distribution in a turbulent boundary layer over a flat plate at length $L$ and free-stream velocity $u_\infty$ is approximated by

$$\frac{u}{u_\infty} = \left(\frac{y}{\delta}\right)^{1/7}.$$ 

Since the wall shear stress cannot be determined with this equation, it is assumed that the local skin-friction coefficient $c_f$ is:

$$c_f = \frac{\tau_W}{\frac{1}{2}u_\infty^2} = \frac{c}{Re_\delta^{1/4}} \quad \text{with} \quad c = 0.046 \quad \text{and} \quad Re_\delta = \frac{\rho u_\infty \delta}{\eta}$$

Determine the distribution of the boundary-layer thickness $\delta(x)$ by the von Kármán integral equation assuming that the flow is fully turbulent at $x \geq 0$.

**Given:** $\rho, u_\infty, \eta, c$

**Hint:**

- von Kármán integral equation:

$$\frac{d\delta_2}{dx} + \frac{1}{u_\alpha} \frac{du_\alpha}{dx} (2\delta_2 + \delta_1) + \frac{\tau(y = 0)}{\rho u_\alpha^2} = 0$$
Problem 6
Flat plate at zero incidence \( \Rightarrow \frac{du_a}{dx} = 0, \; u_a = u_\infty = \text{const.} \)

v. Kármán integral equation:
\[
\frac{d\delta_2}{dx} = -\frac{\tau(y=0)}{\rho u_\infty^2} = \frac{\tau_w}{\rho u_\infty^2} = \frac{1}{2}c_f
\]
\[
\delta_2 = \int_0^\delta \frac{u}{u_\infty} \left(1 - \frac{u}{u_\infty}\right) \, dy = \delta \int_0^1 \frac{u}{u_\infty} \left(1 - \frac{u}{u_\infty}\right) \, dy = \delta \int_0^1 \xi^{1/7} \left(1 - \xi^{1/7}\right) \, d\xi
\]
\[
= \delta \int_0^1 \left(\xi^{1/7} - \xi^{2/7}\right) \, d\xi = \delta \left[\frac{7}{8} \xi^{8/7} - \frac{7}{9} \xi^{9/7}\right]_0^1 = \frac{7}{72}\delta
\]
\[
\frac{d\delta_2}{dx} = \frac{7}{72} \frac{d\delta}{dx} = \frac{1}{2}c_f = \frac{1}{2} c \frac{c}{Re_\delta^{1/4}} = \frac{1}{2} \left(\frac{c}{\rho u_\infty}\right) \frac{1}{4} \frac{1}{\delta^{1/4}}
\]
\[
\Rightarrow \frac{7}{72} \delta^{1/4} \frac{d\delta}{dx} = \frac{1}{2} c \left(\frac{\rho}{\rho u_\infty}\right) \frac{1}{4} x + K_1
\]

Boundary condition: \( \delta(x = 0) = 0 \) \( \Rightarrow \) \( K_1 = 0 \)

\[
\delta^{5/4} = \frac{45}{4} c \left(\frac{\rho}{\rho u_\infty}\right) \frac{1}{4} x
\]
\[
\delta(x) = \left(\frac{45}{4} c\right)^{4/5} \left(\frac{\rho}{\rho u_\infty}\right)^{1/5} x^{4/5}
\]