Exercise 10

1. The Laplace equation
\[ \nabla \cdot \vec{f} = 0 \quad \text{with} \quad \vec{f} = \nabla u \]
is to be solved on a curvilinear structured grid.

(a) Transform the equation for \( \vec{f} \) into curvilinear coordinates \((x, y) \rightarrow (\xi, \eta)\) (conservative form) and discretize the equation for an equidistant grid in curvilinear space.

(b) Formulate a discretization based on a finite volume method for the solution of the equation for \( \vec{f} \). Reformulate the equation as a surface integral, define a meaningful control volume and discretize the equation.

(c) Show that the formulation obtained with the transformation in curvilinear coordinates is identical to the finite volume formulation.
Computational Fluid Dynamics I

Exercise 10 (solution)

1. (a)\[ \nabla \cdot \vec{f} = 0 \quad \vec{f} = \nabla u = \begin{pmatrix} u_x \\ u_y \end{pmatrix} = \begin{pmatrix} g \\ h \end{pmatrix} \]

\[ (x, y) \Rightarrow (\xi, \eta) \quad \nabla \cdot \vec{f} = g_x + h_y = 0 \]

with

\[ g_x = \xi_x g_\xi + \eta_x g_\eta \]
\[ h_y = \xi_y h_\xi + \eta_y h_\eta \]

follows for the terms in the square brackets

\[ \xi_x g_\xi + \eta_x g_\eta + \xi_y h_\xi + \eta_y h_\eta = 0 \quad \mid \cdot J \]
\[ J \xi_x g_\xi + J \eta_x g_\eta + J \xi_y h_\xi + J \eta_y h_\eta = 0 \]

product rule

\[ \frac{\partial}{\partial \xi} (J \xi_x g + J \xi_y h) + \frac{\partial}{\partial \eta} (J \eta_x g + J \eta_y h) - g \left[ \frac{\partial}{\partial \xi} (J \xi_x) + \frac{\partial}{\partial \eta} (J \eta_x) \right] - h \left[ \frac{\partial}{\partial \xi} (J \xi_y) + \frac{\partial}{\partial \eta} (J \eta_y) \right] = 0 \]

with metric terms

\[ \xi_x = \frac{y_\eta}{J}, \quad \xi_y = -\frac{x_\eta}{J}, \quad \eta_x = -\frac{y_\xi}{J}, \quad \eta_y = \frac{x_\xi}{J} \]

follows

\[ \frac{\partial}{\partial \xi} (J \xi_x) + \frac{\partial}{\partial \eta} (J \eta_x) = -\frac{\partial}{\partial \xi} x_\eta + \frac{\partial}{\partial \eta} y_\xi = 0 \]
\[ \frac{\partial}{\partial \xi} (J \xi_y) + \frac{\partial}{\partial \eta} (J \eta_y) = -\frac{\partial}{\partial \xi} y_\eta + \frac{\partial}{\partial \eta} x_\xi = 0 \]

final formulation in curvilinear coordinates

\[ [J(\xi_x g + \xi_y h)]_\xi + [J(\eta_x g + \eta_y h)]_\eta = (y_\eta g - x_\eta h)_\xi + (-y_\xi g + x_\xi h)_\eta = 0 \]

discretisation

\[ (y_\eta g - x_\eta h)_{i+\frac{1}{2},j} - (y_\eta g - x_\eta h)_{i-\frac{1}{2},j} + (-y_\xi g + x_\xi h)_{i,j+\frac{1}{2}} - (-y_\xi g + x_\xi h)_{i,j-\frac{1}{2}} = 0 \]
procedure for the computation (example) for an element

\[ y_\eta g = y_\eta u_x \rightarrow (y_\eta u_x)_{i+\frac{1}{2},j} = (y_\eta)_{i+\frac{1}{2},j} \cdot (\xi_x u_\xi + \eta_x u_\eta)_{i+\frac{1}{2},j} \]

For this we need the metric terms at the point \( i + \frac{1}{2},j \), we can compute these for example by second-order accurate central differences (other formulations possible)

\[ y_{\eta,i+\frac{1}{2},j} = \frac{y_B - y_A}{\Delta \eta} = \frac{y_B - y_A}{1} \]

where \( y_A \) and \( y_B \) are the averages of the surrounding 4 grid points

\[ y_A = \frac{1}{4} (y_{i,j} + y_{i+1,j} + y_{i,j-1} + y_{i+1,j-1}) \]
\[ y_B = \frac{1}{4} (y_{i,j} + y_{i+1,j} + y_{i,j+1} + y_{i+1,j+1}) \]

The other metric terms, e.g., \( \xi_x, \eta_x \), etc, can also be first transformed to the inverse metric terms and then be discretized at \( i + \frac{1}{2},j \) in a similar manner. The terms \( u_\xi \) and \( u_\eta \) can be computed as simple central differences on the computational mesh, e.g.

\[ u_{\xi,i+\frac{1}{2},j} = \frac{u_{i+1,j} - u_{i,j}}{1} \]

(b) finite volume formulation

\[ \int_\tau \nabla \cdot \vec{f} d\tau = \oint_{\partial A} \vec{f} \cdot \vec{n} dA \]
\[ \vec{f} = \begin{pmatrix} g \\ h \end{pmatrix} \]
\[ \vec{n} dA = \begin{pmatrix} dy \\ -dx \end{pmatrix} \]

\[ \rightarrow \int_A gdy - hdx = 0 \]

Possible discretization with node-centered formulation (for mathematical positive direction)
\[
\begin{aligned}
(g \Delta y)_{i+\frac{1}{2},j} &- (h \Delta x)_{i+\frac{1}{2},j} + (g \Delta y)_{i,j+\frac{1}{2}} - (h \Delta x)_{i,j+\frac{1}{2}} \\
+ (g \Delta y)_{i-\frac{1}{2},j} &- (h \Delta x)_{i-\frac{1}{2},j} + (g \Delta y)_{i,j-\frac{1}{2}} - (h \Delta x)_{i,j-\frac{1}{2}} = 0
\end{aligned}
\]

where the corresponding signs (+ for flux entering the volume, − for flux leaving the volume) are contained in the \( \Delta \) terms:

\[
\begin{aligned}
\Delta x_{i+\frac{1}{2},j} &= x_B - x_A \\
\Delta x_{i-\frac{1}{2},j} &= x_D - x_C \\
\Delta x_{i,j+\frac{1}{2}} &= x_C - x_B \\
\Delta x_{i,j-\frac{1}{2}} &= x_A - x_D
\end{aligned}
\]

\[
\begin{aligned}
\Delta y_{i+\frac{1}{2},j} &= y_B - y_A \\
\Delta y_{i-\frac{1}{2},j} &= y_D - y_C \\
\Delta y_{i,j+\frac{1}{2}} &= y_C - y_B \\
\Delta y_{i,j-\frac{1}{2}} &= y_A - y_D
\end{aligned}
\]

give the surface over which the flux is integrated and the correct sign. The coordinates at points \( A, B, C, \) and \( D \) are computed by averages of the surrounding four grid points, as shown before.

(c) curvilinear form

\[
\begin{aligned}
(y_\eta \cdot g)_{i+\frac{1}{2},j} &- (x_\eta \cdot h)_{i+\frac{1}{2},j} - (y_\eta \cdot g)_{i-\frac{1}{2},j} + (x_\eta \cdot h)_{i-\frac{1}{2},j} \\
-(y_\xi \cdot g)_{i,j+\frac{1}{2}} + (x_\xi \cdot h)_{i,j+\frac{1}{2}} + (y_\xi \cdot g)_{i,j-\frac{1}{2}} - (x_\xi \cdot h)_{i,j-\frac{1}{2}} \quad = 0
\end{aligned}
\]

finite volume formulation

\[
\begin{aligned}
(\Delta y \cdot g)_{i+\frac{1}{2},j} &- (\Delta x \cdot h)_{i+\frac{1}{2},j} + (\Delta y \cdot g)_{i-\frac{1}{2},j} - (\Delta x \cdot h)_{i-\frac{1}{2},j} \\
+(\Delta y \cdot g)_{i,j+\frac{1}{2}} &- (\Delta x \cdot h)_{i,j+\frac{1}{2}} + (\Delta y \cdot g)_{i,j-\frac{1}{2}} - (\Delta x \cdot h)_{i,j-\frac{1}{2}} \quad = 0
\end{aligned}
\]

the metric coefficients, e.g., \( x_\eta, y_\xi, \) etc, are then equal to the lengths from the finite volume approach \( \Delta x \) and \( \Delta y \). For example for surface \( i+\frac{1}{2}, j \) we have the metric terms
\[ x_{\eta,i+\frac{1}{2},j} = \frac{x_B - x_A}{\Delta \eta} = \frac{\Delta x_{i+\frac{1}{2},j}}{1} \]
\[ y_{\eta,i+\frac{1}{2},j} = \frac{y_B - y_A}{\Delta \eta} = \frac{\Delta y_{i+\frac{1}{2},j}}{1} \]

The opposite signs in eqs. 1 and 2 are caused by opposite signs in metric terms in comparison with the lengths, for example

\[ -(y_\eta \cdot g)_{i-\frac{1}{2},j} = -\frac{y_C - y_D}{\Delta \eta} (g)_{i-\frac{1}{2},j} = \frac{y_D - y_C}{\Delta \eta} (g)_{i-\frac{1}{2},j} = \Delta y_{i-\frac{1}{2},j} (g)_{i-\frac{1}{2},j} \]

as we compute the metric terms going into positive \( \xi \) and \( \eta \) direction, but for the lengths in the finite volume approach we follow the surface in positive rotation direction, here counterclockwise.