Special problems of Fluid Mechanics
(Experimental analysis of turbulent flows)

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Motivation
&
Contents of the course
Why should engineers study turbulence?

- Most technologically important flows are turbulent
  - Weather forecasts
  - Distribution of hazardous substances in the atmosphere
    - e.g. from accidents in production plants
  - Eruptions of volcanoes -> particles in atmosphere -> disruption of airtraffic etc.
    - e.g. Eyjafjallajökull in 2010
  - Process engineering: heat exchangers, heat and fluid exchange, mixing processes...
    - e.g. injections of jets of fuel in engines
  - Transportation: flows around automobiles, ships, airplanes, hypersonic planes...
    - e.g. aerodynamic stall due to flow phenomena on airfoils
    - fuel consumption is affected
    - excitation of resonant frequencies
  - Propulsion: air-breathing engines, rocket-engine nozzles, propulsive jets...
Why should engineers study turbulence?

• Most technologically important flows are turbulent
  - Turbo machines/fluid-kinetic machines (Turbines, pumps, etc.)
  - Wind energy: Offshore wind parks
    - wake of wind turbines affects turbines in subsequent rows
  - Levees/dams, locks etc.
  - ...
Why should engineers study turbulence?

• Most technologically important flows are turbulent
• Even if the nature of turbulence is not yet completely understood, its effects in real life are too important to wait for the big picture to fall into place.
  – Engineers have to provide solutions to real problems.

Just do not fall into the trap, as William K. George puts it:

“The great danger we face is of being deceived by the successes and good fortune of our “engineering solutions" into thinking we really understand the “physics". But the real world has a way of shocking us back to reality when our “tried and tested" engineering model fails miserably on a completely new problem for which we have not calibrated it. This is what happens when we really don't understand the “physics" behind what we are doing.”

→ Both research and application/engineering have an interest in studying turbulence
What is turbulence?

- State of fluid motion
  - characterized by apparently random and chaotic three-dimensional vorticity

- An exact definition is difficult – but we know some typical properties and features of turbulent flow

- Has the ability to generate new vorticity from old vorticity
- Causes increased energy dissipation, mixing, heat transfer, and drag

- Not entirely chaotic:
  - turbulent flows are time-dependent and space dependent
  - large-scale structures in turbulent flows

- Characteristic frequencies can induce hazardous structural vibrations
What is turbulence?

- Typical features:
  - Spatial and temporal intermittency
  - Unsteadiness
  - Diffusion (strong mixing)
  - Coherent structures
  - Energy is transferred from larger to smaller structures
  - Dissipation (viscosity effects converse kinetic energy in the flow into heat)
    → Turbulent motion decays if there is no external source of energy to continuously generate turbulent motion
  - Near-fractal distribution of scales
  - Instantaneous motions dependent on the initial and upstream conditions
Scope: Gain the knowledge engineers need to have about turbulence

• You will know:
  – Where turbulent flows occur in (aerospace) engineering applications
  – The typical properties of turbulent flows
  – Phenomena related to turbulent flows
  – Which effects and side effects can occur
  – Relevance of turbulent flows for the design process
  – The theoretical background and fundamental equations

• You will be familiar with:
  – Simplifications of the fundamental equations for flows in typical application geometries (as well as the limitations of those simplifications)
  – Experimental methods for turbulence measurements
    – Advantages and drawbacks of the common measurement techniques
  – Modern post-processing techniques for flows with turbulent structures

• You will understand:
  – The processes in turbulent flows relevant for aerospace engineering applications
  – Implications of (geometric and other) conditions on a turbulent flow
Scope: Gain the knowledge engineers need to have about turbulence

• You will be capable to
  – Select applicable simplifications of the fundamental equations suitable for a specific application
  – Judge and select measurement methodologies adequate for a specific case
  – Choose appropriate measurement conditions
  – Analyze (experimental) data – and judge the quality of the data
  – Process and present data so that they can be used by others e.g. for the exchange between experiment and numerical simulations
Contents of the course

• Introduction to turbulence
  - Fundamentals and characteristics
  - Origins and development of turbulence
  - Taylor’s hypothesis
  - Characteristic length and velocity scales
    - Kolmogoroff
    - Taylor microscale

• Governing equations
  - Reynolds-averaged equations
  - Idea of turbulence modelling

• Energy transport in turbulent flows
  - Turbulent energy cascade
Experimental analysis of turbulent flows

- Typical geometries for turbulent flows in engineering
  - Channel flow
  - Developing flows:
    Boundary layers, shear layers, free turbulent flows, such as wake flows and jets
  - Complex flow fields: Shock wave / turbulent boundary layer interactions

- Measurement techniques for turbulent flows
  - Hot-wire anemometry
  - Flow visualization
  - Particle Image Velocimetry
  - Laser Doppler Anemometry
  - Planning an experiment

- Data analysis and error estimation for turbulence measurements
  - Statistics (Fundamentals and definitions, correlations)
  - Errors and uncertainties in measurement data, error propagation
  - Validation and design of experiments
  - Spectral analysis, power spectral density
Contents of the course (continued)

• Common post-processing schemes for turbulent flow data
  - Extraction of turbulent coherent structures from experimental data
  - Proper Orthogonal Decomposition (POD)
  - Dynamic mode decomposition (DMD)
  - Typical applications in fluid dynamics

• Important concepts in turbulence theory
  - Isotropic turbulence
  - Homogeneous turbulence

• Quick overview on numerical methods
Contents of the course (continued)

- Chapter 1: Introduction to turbulence
- Chapter 2: Turbulent flows in engineering
- Chapter 3: The Reynolds Averaged Equations of fluid motion
- Chapter 4: Energy transport in turbulent flows
- Chapter 5: Turbulence measurements
- Chapter 6: Statistical methods
- Chapter 7: Common post-processing schemes for turbulent flow data
- Chapter 8: A few important concepts in turbulence
References & Literature

• William K. George: Lectures in Turbulence for the 21st Century, Lecture notes

• J. Hinze: Turbulence, McGraw-Hill New York


• Hugh W. Coleman, W. Glenn Steele: Experimentation, Validation, and Uncertainty

• Scientific journal papers and chapters of books where appropriate (will be announced)
Chapter 1:
Introduction to turbulence
Transition of boundary layers

- Observation:
  In most technical applications, boundary layers transition from laminar to turbulent.
Experimental analysis of turbulent flows

External disturbances

Amplitude

Receptivity

Transient growth

Primary instability

Secondary instability

Bypass transition

Breakdown

Turbulence
Transition to turbulence on a flat plate (Schematic mechanisms)

External disturbances

Amplitude

Receptivity

Transient growth

(A)
Natural transition
→ Linear stability theory

Primary instability

Secondary instability

Breakdown

Turbulence

Bypass transition
Transition to turbulence on a flat plate (Schematic mechanisms)

- **Natural transition**
  - Laminar base flow
  - Infinitesimally small disturbances

1) Stable laminar flow:
   - all disturbances are damped
2) Primary instability:
   - growth of 2D disturbances,
     formation of T-S-waves
3) Secondary instability:
   - growth of 3D disturbances,
     formation of $\Lambda$-vortices
     (longitudinal vortices)
4) Non-linear growth, formation
   - of turbulent spots, intermittent BL
     (partly laminar, partly turbulent)
   - $Re > Re_{krit}$: shape factor $H_{12} \downarrow$, wall shear stress $\tau_w \uparrow$
5) Fully-turbulent boundary layer

(adapted from Hirschel, Lecture notes (1979))
Transition to turbulence on a flat plate (Schematic mechanisms)

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[adapted from Hirschel, Lecture notes (1979)]
Transition to turbulence on a flat plate (Schematic mechanisms)

- **Bypass-transition**
  - disturbances to base flow not small
  - \( \rightarrow \) non-linear terms in the momentum equation for fluctuations not negligible

Quicker growth of disturbances,

Skipping of steps 2) and 3) in the transition process possible
Experimental evidence for Tollmien-Schlichting waves

- Proof of naturally occurring TS waves
  - Schubauer & Skramstad (1947)
  - Flat-plate boundary layer
  - Low turbulence level wind tunnel: $Tu = 0.3\%$
    \[\rightarrow\] TS waves detectable
  - Hot-wire measurements
    - Oscillating deviation of the instantaneous velocity from temporal mean value
    - Amplitude: < 1\% of inflow velocity

- Visualization of Tollmien-Schlichting waves
  - H. Werlé, ONERA
  - Laminar boundary layer, water tunnel
• T-S waves can be interpreted as spanwise vortices

• T-S waves ... spanwise vortices, superimposed on base flow → Growing disturbance
Origins and development of turbulence

- Turbulent flows arise from laminar flows if (the)
  - Reynolds number increases
  - Turbulence level in the outer flow increases
  - Flow is decelerated (adverse pressure gradient)
  - Flow is separated
  - Surface roughness is increased
    - Generated by friction forces at the rough wall
  - Streamline curvature is increased
  - Base flow is three-dimensional (Cross flow instability)
  - Film cooling is applied (air jets injected through surface)
  - Sound waves interact with leading edge roughnesses

- Small disturbances to the flow are no longer damped by the flow, but begin to grow by taking energy from the original laminar flow

[Transition to turbulence (and Linear Stability Theory) are the subjects of other courses taught at RWTH and will not be covered here in detail.]
What is turbulence?

- Turbulent flow:
  - Fluctuating disturbance motion is superimposed on the mean main flow

- Transient motion is three-dimensional
  - Vortical structures distribute velocity fluctuations in all three spatial dimensions

- Behavior is apparently random and resembles chaos
  → Statistical techniques required

- Turbulent energy mainly contained in larger vortices
  - Energy continuously taken from main flow

- Transfer of energy to increasingly small vortices
  → Kinetic energy cascade due to local energy transfers
  → Decay of larger eddies into smaller vortices

- Smallest vortices: dissipation of turbulent energy to heat
What are the effects of turbulence?

- Causes increased energy dissipation, mixing, heat transfer, and drag

- Turbulent viscosity – an effect created by turbulent diffusion
  - Pressure loss in ducts higher in turbulent than in laminar flow

- In wall-bounded flows:
  - Increased friction forces → Increase in friction drag
  - Increased exchange in momentum
    → more energy in near-wall flow
    → increased heating of surface
    → turbulent boundary layer can overcome larger pressure increases than laminar BL without separating

- Reynolds stress – appears in the averaged equations of motion
  - fluctuating contribution to the non-linear acceleration terms
  - acts as if it were a stress (from the point of view of the mean motion)
  - cannot be related directly to other flow properties (unlike the viscous stress)

  → closure problem
What are the effects of turbulence?

- **Turbulence closure problem:**
  
  There are always fewer equations than unknowns when attempting to predict something more than the instantaneous motions.
  (i.e. the averaged equations are not closed)

  (Not a problem with Direct Numerical Simulations (DNS), where the instantaneous motions are numerically reproduced using the exact governing equations.

  However:
  - DNS for real engineering problems is still too expensive in computational time
  - Even for simple flows the amount of data with apparently random behavior is overwhelming)
Turbulent flow

- Thought experiment:
  - Pressure vessel filled with compressed gas
  - Opening of valve → Gas flows out
  - The flow velocity during this process is measured at location \( r \)

- Laminar flow: velocity curve over time is the same for each realization
- Turbulent flow: Instabilities and variations in the initial and boundary conditions will lead to a different velocity curve \( u_i(r,t) \) for each realization \( i \).

→ Turbulence is a stochastic process
Thought experiment:

- **Turbulence is a stochastic process**

- The question is not: What is the flow velocity $u$ at time $t$ and location $r$?
  - But rather: What is the probability that the velocity has a value between $u$ and $u+\Delta u$ at time $t$?

→ **Turbulence has to be described with statistical quantities:** mean velocity, mean turbulence intensity, probability distributions, characteristic length scales, correlations, spectra...
Typical scales in turbulent flows

- Wide range of characteristic scales in turbulent flows possible
  - Depending on type and topology of flow in question
  - Play an important role for energy dissipation in flows
  - Demanding requirements on spatial and temporal resolutions of measurement techniques

- Energy is transferred from larger to smaller scales:
  - Energy from the base flow is transported into the turbulence through the Reynolds stress working against the mean velocity gradient
  - Dissipation (of turbulence energy) takes kinetic energy from the turbulence and transfers it from larger to smaller scales

- Reynolds stresses are mostly associated with the large and energetic scales of motion
  - Structures scale with the thickness of the shear layer

- Dissipation takes place at the very smallest scales of motion
  - Characterized by the Kolmogorov microscale (in high Reynolds number turbulence)
Turbulence energy cascade

Mean flow \( \ell \) \( \eta \)

Turbulence production \( \rightarrow \) Energy transfer \( \rightarrow \) Dissipation
Typical scales in turbulent flows

• Large eddies – contribute most to the transport of momentum
• Viscous terms prevent the generation of infinitesimally small scales of motion by dissipating small-scale energy into heat

• Turbulence is the primary mechanism for energy dissipation in flows
  – What is the mechanism of energy transfer from large energetic scales to smaller scales?

[We leave this question for later and only define the typical scales for now.]

Energy cascade easy to observe in nature:
  – Add cream into a coffee cup
  – Observe evolution from large scales to small scales during mixing process

• Length scales:
  • Large-scale eddies: “diffusive length scale” \( l \)
    – Represents width/thickness of the shear/boundary layer
  • Length \( L \) of the shear flow: “convective length scale”

• Time scales:
  • Convective time scale \( L/U \)
  • Diffusive time scale \( l/u \)

Turbulent time scale comparable with time scale of the main flow
Typical scales in turbulent flows

- Small-scale motions have small time scales
  - Statistically independent of the relatively slow large-scale turbulence and of the mean flow
  - Small-scale motion depends only on the rate at which it is supplied with energy by the large-scale motion and on the kinematic viscosity.
- Rate of energy supply should be equal to the rate of dissipation
- Kolmogorov's universal equilibrium theory of the small-scale structure
  - Parameters governing the small-scale motion:
    - Dissipation rate per unit mass $\varepsilon$
    - Kinematic viscosity $\nu$

**Kolmogorov microscales** of length, time, and velocity:

$$\eta = \left( \frac{\nu^3}{\varepsilon} \right)^{1/4} \quad \tau = \left( \frac{\nu}{\varepsilon} \right)^{1/2} \quad \nu = \left( \nu \cdot \varepsilon \right)^{1/4}$$

- The Reynolds number formed with $\eta$ and $\nu$ is equal to unity: $\eta \cdot \nu / \nu = 1$
  - Small-scale motion is viscosity dominated
  - Viscous dissipation adjusts itself to the energy supply by adjusting length scales
Experimental analysis of turbulent flows

Chapter 2
Chapter 2:
Turbulent flows in engineering
Typical geometries for turbulent flows in engineering

- Free shear flows:
  - Jets
    - Plane jets
    - Round jet (quiet or moving surroundings)
    - “Half jet” – boundary between two parallel flows with different mean velocities
    - Axial / coaxial jets (two-dimensional, rotationally symmetric)
  - Wake flows
    - Plane wake behind cylindrical rod
    - Round wake behind axisymmetric body

- Wall-bound flows:
  - Boundary layers
  - Pressure gradient boundary layers
  - Channel flow
  - Pipe flow

- Complex flow fields:
  - Shock wave / turbulent boundary layer interactions
Free shear flows – Jets and wakes

- **Wake**: Turbulent flow field developing behind an obstacle / body placed in an incoming flow

- **Jet**: Stream of fluid issued from an orifice / nozzle into a quiet medium (or into a moving medium)

[adapted from Tennekes and Lumley, A first course in turbulence, MIT Press, 1972]
• Occur in nature in many different forms and scales:
  – Exhaling: jet of air from nostrils or mouth
  – Plume from a cigarette
  – Volcano eruption: buoyant jet
  – ...

• Examples for jet flows in engineering:
  – Exhausts from jet engines, rocket engines...
  – Most combustion processes
  – Mixing processes

• Examples for wake flows in engineering:
  – Wake behind moving vehicles (cars, planes, ...)

Important difference between jets and wakes:
• Jet: at the source, momentum is being added to the flow
• Wake: momentum is being taken out (by drag)
Features of free shear flows

• Inhomogeneous

• Mean velocity gradients develop in the absence of boundaries

• Most often turbulent or transitioning to turbulence very quickly:
  – Transition to turbulence requires three-dimensional vorticity
  – Can develop rapidly if there are no walls that inhibit the growth of velocity components normal to them.

• Vortical fluid is spatially confined

• Vortical fluid is separated from the surrounding fluid by an interface:
  – Turbulent – non-turbulent interface ("Corrsin superlayer")
  – Thickness: characterized by the Kolmogorov microscale, thus its characterization as an interface is appropriate.
  – Shape: random, and severely distorted by the energetic turbulent processes in the vortical fluid
    → State can be highly intermittent

[ Intermittent = at a given location, the flow is sometimes turbulent, and sometimes not ]
Example: Round jet

- Velocity vs. time traces at two locations in the mixing layer of a round jet
  - Center of mixing layer: maximum shear
  - Outer edge of shear layer

[adapted from William K. George: Lectures in Turbulence for the 21st Century, Lecture notes]
Round jet

• Velocity vs. time traces at two locations in the mixing layer of a round jet

• Turbulent fluid passes the probes → signals characterized by bursts of activity

• Smooth zones between bursts:
  irrotational fluctuations induced by vorticity on turbulent side of the interface

• Center of the mixing layer: maximum shear
  – Maximum production of turbulent energy
    → Flow nearly always turbulent

• Further away from center: increasingly intermittent
  – Outside of zone of maximum turbulence energy production
  – Turbulent/non-turbulent interface constantly changing its position
    → Characteristic of all free shear flows

Gradients in mean velocity and mean shear stresses → turbulence
Features of free shear flows

- Amount of turbulent fluid is continuously increased by **entrainment**
  - Turbulent part of the flow continues to capture additional fluid from surroundings
  - Mass flow of the jet increases at each cross-section

→ Free shear flow continues to spread
→ Flow becomes more and more diluted by addition of outside fluid
  - basis of many technical mixing processes
→ Turbulent transport terms in equations of motion cannot be neglected
  (at least in the direction(s) in which the flow is spreading)
Features of free shear flows

- Coherent large scale structures or eddies develop
  - Scale equal to lateral extent of the flow
  - Scales continue to grow while turbulent shear flow develops
  - Shape of turbulent/non-turbulent interface dominated by large-scale structures

- Role in entrainment process:
  - Large structures stretch the small scale vorticity on the interface
    → amplification of small-scale vortical structures
    → vorticity diffuses rapidly into the non-turbulent fluid
  - Participate in processes by which turbulence gains and distributes energy from the mean flow

Very nice visualization:
Turbulence in a water jet
Features of free shear flows

• In all the flow patterns mentioned, there is one main flow direction in which the velocity is much greater than in any other direction
  – Allows simplification of the equations

• Assumptions that can be made:
  (1) Mean flow velocity transverse to main flow can be assumed to be very small
    – In some cases even negligible
  (2) Changes of quantities in the main flow direction are slow compared to changes in the transverse direction
  (3) Mean-pressure variation across the shear flow
    – In the transverse direction:
      mainly determined by variation in the turbulent velocity intensities
    – In the main-flow direction:
      mainly determined by pressure distribution in undisturbed outer flow

Often: uniform pressure distribution in the main-flow direction
→ Assumption to calculate mean-velocity distribution:
  mean pressure is uniform throughout turbulent region
Similarity and free shear flows

- Is there similarity / self-preservation in free shear flows?

- **Self-preservation** – turbulence structure is maintained during the development of the turbulent flow (in the downstream direction)

- In jet and wake flows, mean velocity profiles are similar in the direction of development.
  - Have to be made dimensionless:
    - Velocity scale: maximum mean-velocity difference
    - Length scale: Lateral/radial distance normalized with “half-value” distance
      - Both the velocity and length scales are functions of the distance from the origin of the jet or wake flow – and only of that.

- How is this possible?
  - Main flow velocity >> transverse velocity
  - Spatial changes in main-flow direction << changes in transverse direction
    → Turbulence flow pattern strongly dependent on its history
Similarity and free shear flows

• Without continuous turbulence production in the shear layer, turbulence would decay while convected downstream
  
  – Gradient of mean-velocity distribution
  – Mean shear stresses

→ Close connection between turbulence and mean-velocity distribution
  
  → Similarity of turbulence can be expected

• The flow field needs to fully develop before the velocity profiles become self-preserving
Example: The plane turbulent jet

- Jet issues from an orifice into a quiet medium

[adapted from Tennekes and Lumley, A first course in turbulence, MIT Press, 1972]
Example: The plane turbulent jet

- Fluid issues from an orifice into a quiet medium

- Stages of jet development:
  - Two plane mixing layers, separated by a core of irrotational flow
  - Mixing layers merge
  - Fully developed, self-preserving turbulent flow

- Why are we interested in the fact whether or not a flow is self-preserving?
  - It allows to reduce the number of independent variables by one

- Helpful aspect: Wakes and jets have one main flow direction

→ Application of boundary-layer approximations for equations possible
The boundary-layer approximation [...] will lead to the shear-layer equations

- Approximating assumptions to simplify the Navier-Stokes equations
  - Originally proposed by Prandtl for wall boundary layers
  - Central idea: different length scales for changes perpendicular to the wall and in the main flow direction
  - Basic approximations can be applied to all slowly growing shear flows with or without a surface

- The boundary-layer approximation is here presented for flows that are:
  - Plane (or two-dimensional) in the mean
  - Having a main-flow direction (x-direction)
  - Slowly evolving in the main-flow direction $\partial/\partial x \ll \partial/\partial y$
  - Statistically stationary $\rightarrow$ time derivatives of averaged quantities negligible
  - Similar considerations can be applied to axisymmetric flows
  - Also: incompressible, Newtonian fluid

→ We consider the main flow direction $x$ and one transverse direction $y$
  - $U$ and $u$ ... mean and fluctuating velocities in the main-flow direction $x$
  - $V$ and $v$ ... mean and fluctuating velocities in the transverse direction $y$
The boundary-layer approximation

• The mean momentum equations are then:
  
  – In the main-flow direction $x$:
    \[
    U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial x} - \frac{\partial \langle u^2 \rangle}{\partial x} - \frac{\partial \langle uv \rangle}{\partial y} + \nu \frac{\partial^2 U}{\partial x^2} + \nu \frac{\partial^2 U}{\partial y^2}
    \]
  
  – In the transverse direction $y$:
    \[
    U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial y} - \frac{\partial \langle uv \rangle}{\partial x} - \frac{\partial \langle v^2 \rangle}{\partial y} + \nu \frac{\partial^2 V}{\partial x^2} + \nu \frac{\partial^2 V}{\partial y^2}
    \]

• Mean continuity equation:
  \[
  \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0
  \]

• To simplify the equations, an estimate for the order of magnitude for each of the terms in the equations has to be made.
Estimates for the orders of magnitude of terms

• **Step 1:** Choose different length scales for changes in the x- and y-directions:
  - Length scale characteristic of quantities in the main-flow direction:
    \[
    \frac{\partial}{\partial x} \sim \frac{1}{L}
    \]
  - Length scale characteristic of quantities in the direction normal to the mean flow:
    \[
    \frac{\partial}{\partial y} \sim \frac{1}{\delta}
    \]

    These scales are a function of the streamwise position where the quantities are evaluated!

• The choice for the specific length scale depends on the flow. Typical choices:
  - \( L \) ... proportional to the distance from the source / origin of the shear flow
  - \( \delta \) ... proportional to the local width / height of the flow
• **Step 2:** Velocity scales for velocity component U in the main-flow direction

• In the governing equations, the mean velocity U occurs in different terms:

\[
\bar{U} \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial x} - \frac{\partial \langle u^2 \rangle}{\partial x} - \frac{\partial \langle uv \rangle}{\partial y} + \nu \frac{\partial^2 U}{\partial x^2} + \nu \frac{\partial^2 U}{\partial y^2}
\]

  - Alone, with x-derivatives, and with y-derivatives

• Is there one scale velocity that can be used to estimate all those terms?
  
  - This, of course, depends on the flow field that we want to investigate.
    
    (We will see that in the following example.)
Estimates for the orders of magnitude of terms – Length scales

- **Jet**
  - **x-direction:** Local distance $L$ from orifice
  - **y-direction:** Half-value distance $l$

- **Wake**
  - **x-direction:** Distance $L$ from wake generator
  - **y-direction:** Half-value distance $l$

**Half-value distance** = distance from axis of symmetry at which mean-velocity difference is half the maximum value.
Estimates for the orders of magnitude of terms – Velocity scales

- Velocity scale:
  - Mean centerline velocity $U_s(x)$ at a given streamwise location $x$
  - Characterizes both the mean velocity $U(x,y)$ and changes of $U$

- Wake deficit $\ll$ free-stream velocity (even close to wake generator)
  - $U_0$ is appropriate scale for $U(x,y)$
  - $U_0/\delta$ would be much too large as an estimate for the velocity gradient
  - Centerline mean velocity deficit $U_s$

\[
\frac{\partial U}{\partial y} \sim \frac{U_s}{\delta}
\]
The “context” is important for turbulent shear flows

• Two general classes of free shear flows:
  – With external flow
  – Quiet surrounding medium

• Turbulent shear flows spread by exchanging momentum with the surrounding flow
  – Often they also entrain mass from the surrounding fluid

State of the surrounding fluid strongly influences the development of the shear flow

• Practical implications when investigating the flow:

  (1) Conditions in surrounding fluid influence which mean convection terms must be kept in the governing equations

  (2) Influence of surrounding fluid has to be taken into account when determining whether an experiment or simulation is a valid approximation to a flow
    – Usually performed in a finite grid / wind tunnel
    – e.g. Flow around an air-plane wing in the atmosphere vs. in a small wind tunnel with a certain turbulence level
Estimates for the orders of magnitude of terms

• **Step 2**: Velocity scales for velocity component U in the main-flow direction

  • More general derivation:
    – $U_s$ to characterize the mean velocity
    – $\Delta U_s$ to represent changes in the mean velocity

• **Step 3**: Scales for changes of the velocity U in the $x$- and $y$-directions

  • Idea:
    – Same estimate for changes of U in the $x$- and $y$-directions
    – Define the length scale $L$ accordingly

\[
\frac{\partial U}{\partial y} \sim \frac{\Delta U_s}{\delta} \quad \text{and} \quad \frac{\partial U}{\partial x} \sim \frac{\Delta U_s}{L}
\]

[In the previous example $\Delta U_s = U_s$ for the jet, while they take different values for the wake because of the surrounding flow.]
Estimates for the orders of magnitude of terms

• **Step 4:** Estimate the velocity scale for the cross-stream mean velocity component $V$

• From the continuity equation we get:

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0$$

$$\frac{\partial V}{\partial y} = -\frac{\partial U}{\partial x}$$

• We also know that

$$\frac{\partial U}{\partial x} \sim \frac{\Delta U_s}{L}$$  
(see previous slide)

• If there is no mean cross flow in the external stream
  – $V$ and changes in $V$ scale with the same quantity

$$\frac{\partial V}{\partial y} \sim \frac{V_s}{\delta}$$

**i.e. external flow has only a velocity component in the shear-flow main direction**
(or: quiet surrounding medium)
Estimates for the orders of magnitude of terms

• **Step 4:** Estimate the velocity scale for the cross-stream mean velocity component $V$

• The order of magnitude of the transverse velocity component is therefore:

$$V_s \sim \frac{\Delta U_s \delta}{L}$$

• **Step 5:** Estimate order of magnitude of the turbulence terms

• Pressure strain-rate terms evenly distribute the energy
  
  $\rightarrow$ Components of the turbulent velocity usually about the same order of magnitude

• Choice of one turbulence scale $u$

  $\rightarrow <u^2> \sim u^2$, $<v^2> \sim u^2$, and $<uv> \sim u^2$

• These are only estimates to find the terms that (probably) are most important most of the time.

  $\rightarrow$ Check necessary after analyzing or solving the equations!
Estimates for the orders of magnitude of terms

- Mean pressure gradient term difficult to estimate without additional information
  → Leave pressure term until the end

- If there is no mean cross flow in the external stream:
  - $V$ and changes in $V$ scale with the same quantity
    (otherwise, similar considerations as for $U$ have to be made)
Estimates for the orders of magnitude of terms

- **Step 6**: Apply defined scales to estimate the importance of terms in the governing equations.

- **x-component of the mean momentum equation**:

\[
U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = \frac{1}{\rho} \frac{\partial P}{\partial x} - \frac{\partial \langle u^2 \rangle}{\partial x} - \frac{\partial \langle uv \rangle}{\partial y} + \nu \frac{\partial^2 U}{\partial x^2} + \nu \frac{\partial^2 U}{\partial y^2}
\]

\[
= U \frac{\Delta U_s}{L} \left( \Delta U_s \frac{\delta}{L} \right) \frac{\Delta U}{\delta} + \text{?} + \frac{u^2}{L} + \frac{u^2}{\delta} + \nu \frac{\Delta U_s}{L^2} + \nu \frac{\Delta U_s}{\delta^2}
\]

- We are looking at free shear flows
  - The flow is either accelerated or decelerated by the external flow or the surrounding quiet fluid.
  - At least one of the terms on the left-hand side of the equation must remain
Step 6: Apply defined scales to estimate the importance of terms in the governing equations

The $x$-component of the mean momentum equation:

\[
\frac{\partial U}{\partial x} U + V \frac{\partial U}{\partial y} = \frac{1}{\rho} \frac{\partial P}{\partial x} - \frac{\partial \langle u^2 \rangle}{\partial x} - \frac{\partial \langle uv \rangle}{\partial y} + \nu \frac{\partial^2 U}{\partial x^2} + \nu \frac{\partial^2 U}{\partial y^2}
\]

\[
U_s \frac{\Delta U_s}{L} \quad \left( \frac{\Delta U_s}{L} \right) \frac{\Delta U}{\delta} \quad ? \quad \frac{u^2}{L} \quad \frac{u^2}{\delta} \quad \frac{\Delta U_s}{L^2} \quad \frac{\Delta U_s}{\delta^2}
\]

- $x$-direction is the main-flow direction
  \[\rightarrow \text{The first term on the LHS must be the largest.}\]
- Assessment of importance of the remaining terms (relative to the most important term):
  - Rescaling of the terms by dividing the estimates by:
    \[\frac{U_s \Delta U_s}{L}\]
Estimates for the orders of magnitude of terms

- **Step 6:** Apply defined scales to estimate the importance of terms in the governing equations

- **x-component of the mean momentum equation:**

\[
U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial x} - \frac{\partial \langle u^2 \rangle}{\partial x} - \frac{\partial \langle uv \rangle}{\partial y} + \nu \frac{\partial^2 U}{\partial x^2} + \nu \frac{\partial^2 U}{\partial y^2}
\]

\[
1 + \frac{\Delta U_s}{U_s} + \frac{u^2}{U_s \Delta U_s} + \frac{u^2}{U_s \Delta U_s} \left( \frac{L}{\delta} \right) + \frac{\nu}{U_s L} + \frac{\nu}{U_s \delta} \left( \frac{L}{\delta} \right)
\]

- **Step 7:** Assessment of importance of the remaining terms by using “engineering judgment”
Estimates for the orders of magnitude of terms

\[
\begin{align*}
U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} &= -\frac{1}{\rho} \frac{\partial P}{\partial x} - \frac{\partial \langle u^2 \rangle}{\partial x} - \frac{\partial \langle uv \rangle}{\partial y} + \nu \frac{\partial^2 U}{\partial x^2} + \nu \frac{\partial^2 U}{\partial y^2} \\
&\approx 1 + \frac{\Delta U_s}{U_s} + \frac{u^2}{U_s \Delta U_s} + \frac{u^2}{U_s \Delta U_s} \left( \frac{L}{\delta} \right) + \frac{\nu}{U_s L} + \frac{\nu}{U_s \delta} \left( \frac{L}{\delta} \right)
\end{align*}
\]

- Most flows of interest for aerospace-engineering applications: at high Reynolds number
  - i.e. inertial forces more important than viscous forces for given flow conditions

\[ Re >> 1 \quad \rightarrow \quad \text{viscous terms negligible} \]

\[ \frac{U_s L}{\nu} >> 1 \quad \text{and} \quad \frac{U_s \delta}{\nu} >> \frac{L}{\delta} \]

Flow around air-plane wings...
The second criterion has to be handled with care:

- We discuss thin shear flows \( \leftrightarrow \frac{L}{\delta} \approx 10 \)
- If \( \frac{U_s \delta}{\nu} > 1000 \): contributions of the viscous stresses to the \( x \)-component of the mean momentum equation are negligible
- If \( \frac{U_s \delta}{\nu} \sim 100 \): contributions of the viscous stresses probably not negligible \( \rightarrow \) term stays in equation

\[ \frac{U_s L}{\nu} \gg 1 \]

\[ \frac{U_s \delta}{\nu} \gg \frac{L}{\delta} \]

\( \rightarrow \) May occur in experiments that cannot be carried out at high Reynolds number.
Estimates for the orders of magnitude of terms

\[
\begin{align*}
U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} &= \frac{-1}{\rho} \frac{\partial P}{\partial x} - \frac{1}{\rho} \frac{\partial \langle u^2 \rangle}{\partial x} - \frac{1}{\rho} \frac{\partial \langle uv \rangle}{\partial y} + \frac{\nu}{\partial x^2} + \frac{\nu}{\partial y^2} \\
1 &+ \frac{\Delta U_s}{U_s} ? \frac{u^2}{U_s \Delta U_s} \frac{u^2}{U_s \Delta U_s} \left( \frac{L}{\delta} \right) \frac{\nu}{U_s L} \frac{\nu}{U_s \delta} \left( \frac{L}{\delta} \right)
\end{align*}
\]

- Information on orders of magnitude from the turbulence terms
  - Usually the fluctuations are smaller than the mean values: \( u < U_s \) (sometimes even much smaller)
  - We consider thin shear layers \( \rightarrow L > \delta \), i.e. \( L/\delta > 1 \)
    - The biggest turbulence term is the one involving the Reynolds shear stress:
      \[
      \frac{u^2}{U_s \Delta U_s} \left( \frac{L}{\delta} \right) \sim 1 \rightarrow \frac{\delta}{L} \sim \frac{u^2}{U_s \Delta U_s}
      \]
      i.e. Growth of a free shear flow can be estimated from the turbulence intensity

At least one important turbulence term required to balance eq.
Estimates for the orders of magnitude of terms

\[
\frac{\partial U}{\partial x} + \frac{V \partial U}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial x} - \frac{\partial \langle u^2 \rangle}{\partial x} - \frac{\partial \langle uv \rangle}{\partial y} + \frac{\nu}{\partial x^2} + \frac{\nu}{\partial y^2}
\]

- Information on orders of magnitude from the turbulence terms
  - \( L/\delta \geq 10 \)

  \[
  \frac{u^2}{U_s \Delta U_s} \left( \frac{L}{\delta} \right) > \frac{u^2}{U_s \Delta U_s}
  \]

  → Turbulence term \( \frac{\partial \langle u^2 \rangle}{\partial x} \) only about 10% of the magnitude of \( \frac{\partial \langle uv \rangle}{\partial y} \)

  → Depending on the required accuracy of the result, the term may be neglected or not
Estimates for the orders of magnitude of terms

\[ \frac{U}{\partial x} + \frac{V}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial x} - \frac{\partial \langle u'^2 \rangle}{\partial x} - \frac{\partial \langle uv \rangle}{\partial y} + \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \]

- Required accuracy of a result is an important factor for approximations in engineering
  - If a second-order statement is required, the equations/data/… used to make this assessment should of course be of second order accuracy!
  - Example: New in-house simulation tool is developed, and a second-order turbulence model is implemented. To validate the code, results are compared with experimental data. The measurements should then also be of second-order accuracy.
- Last Step: Check the results for consistency
Step 6 (continued): Apply defined scales to estimate the importance of terms in the governing equations

- The y-component cannot be considered independently from the x-component!
  - It is a vector equation, so all components have to be rescaled in the same way

- x-direction is the main-flow direction
  - The first term on the LHS of the x-component must be the largest \( U \frac{\partial U}{\partial x} \)
  - So far, relative importance of terms was estimated by comparing it to this term
  - Estimate has to be continued in the same way for the y-component of the equation

\[
U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial y} - \frac{\partial \langle uv \rangle}{\partial x} - \frac{\partial \langle v^2 \rangle}{\partial y} + \nu \frac{\partial^2 V}{\partial x^2} + \nu \frac{\partial^2 V}{\partial y^2}
\]
Estimates for the orders of magnitude of terms

- **Step 6 (continued):** Apply defined scales to estimate the importance of terms in the governing equations.

- **y-component of the mean momentum equation:**
  \[
  U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} = \frac{1}{\rho} \frac{\partial P}{\partial y} - \frac{\partial \langle uv \rangle}{\partial x} - \frac{\partial \langle v^2 \rangle}{\partial y} + \nu \frac{\partial^2 V}{\partial x^2} + \nu \frac{\partial^2 V}{\partial y^2}
  \]

- Assessment of importance of the remaining terms relative to the most important term:
  - Rescaling of the terms by dividing the estimates by: \( U_s \frac{\Delta U_s}{L} \)
Estimates for the orders of magnitude of terms

- **Step 6 (continued):** Apply defined scales to estimate the importance of terms in the governing equations

- **y-component of the mean momentum equation:**

  \[
  \frac{U}{L} \frac{\partial V}{\partial x} + \frac{V}{L} \frac{\partial V}{\partial y} = \frac{1}{\rho} \frac{\partial P}{\partial x} - \frac{\partial \langle uv \rangle}{\partial x} - \frac{\partial \langle v^2 \rangle}{\partial y} + \nu \frac{\partial^2 V}{\partial x^2} + \nu \frac{\partial^2 V}{\partial y^2}
  \]

  - \( \frac{\delta}{L} \)
  - \( \frac{\Delta U_s}{U_s} \left( \frac{\delta}{L} \right) \)
  - \( \frac{\nu}{U_s \Delta U_s} \left( \frac{L}{\delta} \right) \)
  - \( \frac{\nu}{U_s L} \left( \frac{\delta}{L} \right) \)
  - \( \frac{\nu}{U_s \delta} \)

- **Mean convection terms:** not of order one

  \[
  \frac{\delta}{L} \sim 0.1 \quad \frac{\Delta U_s}{U_s} \left( \frac{\delta}{L} \right) \quad \text{... depends on flow}
  \]

  *see wake- and jet-example*
Estimates for the orders of magnitude of terms

- **Step 6 (continued):** Apply defined scales to estimate the importance of terms in the governing equations

  - **y-component of the mean momentum equation:**

    $$ U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} = \frac{1}{\rho} \frac{\partial P}{\partial x} - \frac{\partial \langle uv \rangle}{\partial x} - \frac{\partial \langle v^2 \rangle}{\partial y} + \nu \frac{\partial^2 V}{\partial x^2} + \nu \frac{\partial^2 V}{\partial y^2} $$

    - Viscous terms are negligible since the Reynolds number is usually high for flows that are relevant in aerospace engineering

    - Viscous terms: $\nu \frac{\partial^2 V}{\partial x^2}$ and $\nu \frac{\partial^2 V}{\partial y^2}$

    - $\frac{\delta}{L}$
    - $\frac{\Delta U_s}{U_s} \left( \frac{\delta}{L} \right)$
    - $\frac{u^2}{U_s \Delta U_s}$
    - $\frac{u^2}{U_s \Delta U_s} \left( \frac{L}{\delta} \right)$
    - $\frac{\nu}{U_s L} \frac{\delta}{L}$
    - $\frac{\nu}{U_s \delta}$

      - As $Re >> 1$, $\frac{\nu}{U_s \delta}$ is $\ll 1$
      - As $Re \leq 0.1$, $\frac{\nu}{U_s L}$ is $\leq 0.1$
Estimates for the orders of magnitude of terms

- **Step 6 (continued):** Apply defined scales to estimate the importance of terms in the governing equations

  - **y-component of the mean momentum equation:**

    $U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} = \frac{1}{\rho} \frac{\partial P}{\partial y} - \frac{\partial \langle uv \rangle}{\partial x} - \frac{\partial \langle v^2 \rangle}{\partial y} + \nu \frac{\partial^2 V}{\partial x^2} + \nu \frac{\partial^2 V}{\partial y^2}$

    - Turbulence term $\frac{\partial \langle v^2 \rangle}{\partial y}$ is of order one (as discussed for the Reynolds-shear stress term in the x-component of the equation)
    - Turbulence term $\frac{\partial \langle uv \rangle}{\partial x}$ is about 10% of $\frac{\partial \langle v^2 \rangle}{\partial y}$
Estimates for the orders of magnitude of terms

In first order, the y-component of the mean momentum equation reduces to:

\[ 0 \approx -\frac{1}{\rho} \frac{\partial P}{\partial y} - \frac{\partial \langle v^2 \rangle}{\partial y} \]

This means that the change in the mean pressure is only caused by the radial gradient of the normal Reynolds stress component in the y-direction.
Estimates for the orders of magnitude of terms

• This equation can be integrated and then substituted into the x-component of the momentum equation to eliminate the pressure terms

• \( y \)-component of the mean momentum equation in first order:

\[
0 \approx -\frac{1}{\rho} \frac{\partial P}{\partial y} - \frac{\partial \langle v^2 \rangle}{\partial y}
\]

Integration across shear layer \( \int_y^\infty \)

- Turbulent velocity fluctuations in the free stream are very small
  → can be assumed to be zero

- Surrounding fluid at constant mean velocity: constant mean pressure in free stream

\[
P(x, y) = P_\infty - \rho \langle v^2 \rangle
\]
Estimates for the orders of magnitude of terms

- One more example that the question we want answered is very relevant for the decision on which terms can be neglected:
  \[
  \langle v^2 \rangle \text{ is generally very small compared to velocity changes in the y-direction}
  \]
  \[
  \rightarrow \text{It might be possible to neglect the term: } P(x, y) \approx P_\infty
  \]
  - This would mean that small pressure changes across the flow are not important.

- What is the role of the small pressure changes in the transverse direction?
  - They govern the entrainment of surrounding fluid into the shear layer and therefore the shear-layer growth.
    \[
    \rightarrow \text{Influence shape and characteristics of the flow field}
    \]

(1) We just want to estimate the streamwise momentum
  \[
  \rightarrow \text{Assumption is valid: it allows to answer our question, while keeping the equation as simple as possible}
  \]

(2) The wake (e.g. behind a vehicle) induces pressure fluctuations on the surface of the vehicle. These fluctuations induce structural vibrations that we want to prevent.
  \[
  \rightarrow \text{We need to understand the flow field} \rightarrow \text{Characteristics need to be maintained}
  \]
Estimates for the orders of magnitude of terms

For large Reynolds numbers, the momentum equations for free shear flows reduce to one equation:

- Evaluation of pressure term in the x-momentum equation with derivative with respect to x of:

\[ P(x, y) = P_\infty - \rho \langle v^2 \rangle \]

- Introduction into x-component of the momentum equation:

\[
U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = -\frac{1}{\rho} \frac{d P_\infty}{dx} - \frac{\partial}{\partial y} \langle uv \rangle - \frac{\partial}{\partial x} \left[ \langle u^2 \rangle - \langle v^2 \rangle \right]
\]
For large Reynolds numbers, the momentum equations for free shear flows reduce to one equation:

- Evaluation of pressure term in the x-momentum equation with derivative with respect to x of:
  \[ P(x, y) = P_\infty - \rho \langle v^2 \rangle \]

- Introduction into x-component of the momentum equation:
  \[ U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = -\frac{1}{\rho} \frac{dP_\infty}{dx} - \frac{\partial}{\partial y} \langle uv \rangle - \frac{\partial}{\partial x} \left[ \langle u^2 \rangle - \langle v^2 \rangle \right] \]

Relevance depends on the velocity deficit (or excess) relative to the free stream
Surrounding fluid at constant mean velocity → constant mean pressure in free stream
Second order in turbulence intensity, but does not vanish with distance from source for some flows
The plane jet (resting surrounding medium)

- Stream of fluid issues from a slot-shaped orifice into a medium at rest
- Evolves and spreads downstream by entraining mass from surrounding medium
- Vortical motion in jet induces irrotational motion in surrounding fluid
- Downstream of the orifice, no new momentum is added to the jet

Surrounding medium at rest $\rightarrow$ zero streamwise velocity in free stream $\rightarrow$ Same scales for the velocity and velocity gradients applicable

\[ \frac{\partial}{\partial x} U^2 + \frac{\partial}{\partial y} UV + \frac{\partial}{\partial y} \langle uv \rangle = 0 \]

[adapted from Tennekes and Lumley, A first course in turbulence, MIT Press, 1972]

i.e. governing equations are the resulting momentum and the mean continuity eqs.
The plane jet (resting surrounding medium)

- The mean velocity \( V \) in the \( y \)-direction can be determined by integrating the averaged continuity equation from the centerline to \( y \):

\[
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0
\]

Integration across jet \( \int_{y=0}^{y} \)

\[
V = -\int_{0}^{\infty} \frac{\partial U}{\partial x} d\tilde{y}
\]

Half of the flow-rate across a given \( x \)-plane

- Rate of increase of the integral = entrainment velocity
  i.e. velocity with which fluid from the surrounding medium is transported into the jet

\[
V_\infty = -V_{-\infty} = -\frac{d}{dx} \int_{0}^{\infty} U(x, y) dy
\]

→ Integral of velocity perpendicular to main-flow direction from the axis to \( \infty \)
The plane jet (resting surrounding medium)

- A typical choice of length scale for the transversal direction is the half-value distance $l$, i.e. the distance from the axis of symmetry at which mean-velocity difference is half the maximum value.

- For the main-flow direction, the distance $L$ from the origin of the jet is a typical scale.

- For jets, as well as mixing layers, the ratio $l/L$ of these length scales remains constant along the developing flow.

  Just to get an idea about a realistic order of magnitude for the length scale ratio: in experiments $l/L = O(6\%)$ for jets and mixing layers.

- For wakes on the other hand, $l/L$ continually decreases downstream.

  [See Tennekes and Lumley, A first course in turbulence, MIT Press, 1972, for details.]
The momentum integral

• The momentum integral is obtained by integrating the appropriate momentum equation from the source of the shear flow (in the x-direction)

• For the plane jet this leads to an expression of the form:

\[ M_o = \int_{-\infty}^{\infty} \left( U^2 + \left( \langle u^2 \rangle - \langle v^2 \rangle \right) \right) dy \]

\( M_o \) is a very useful expression to

- Determine a similarity solution for the flow in question (together with the corresponding governing equations)
- Applicable to check experimental data on validity
- Usable to define a length scale for turbulent wakes, the momentum thickness of the wake (which is related to the drag coefficient).
Similarity solution for the plane jet

• Similarity solutions to the averaged equations of motion are of the form:

\[
\begin{align*}
U &= U_s(x)f(\bar{y}, *) \\
\langle uv \rangle &= R_s(x)g(\bar{y}, *)
\end{align*}
\]

where \(U_s\) and \(R_s\) are characteristic scales,
\(f\) and \(g\) are scale functions depending on
\[
\bar{y} = \frac{y}{\delta(x)}
\]
and – possibly – the upstream conditions, indicated by \(*\).

• Next: Introduce formulation for similarity solutions into averaged equations and assess whether or not a solution in the desired form exists,
i.e. we search a solution for which the important terms in the equations of motion remain equally important throughout the jet development
(equilibrium similarity solutions)
Similarity solutions to the averaged equations of motion are of the form:

\[ U = U_s(x) f(\overline{y}, \cdot) \]

\[ -\langle uv \rangle = R_s(x) g(\overline{y}, \cdot) \]

where \( U_s \) and \( R_s \) are characteristic scales, \( f \) and \( g \) are scale functions depending on \( \overline{y} = y / \delta(x) \) and – possibly – the upstream conditions, indicated by \( \cdot \).

Next: Introduce formulation for similarity solutions into averaged equations and assess whether or not a solution in the desired form exists,

i.e. we search a solution for which the important terms in the equations of motion remain equally important throughout the jet development (equilibrium similarity solutions)

We will look at that in more detail in the tutorial.
The turbulent wake (far from the obstacle)

- Wake deficit $U_0 - U \ll$ free-stream velocity $U$ (even close to wake generator)
  - $U_s = U_0$ is appropriate scale for $U(x,y)$
  - $\Delta U_s = U_0 - U$ good scale for changes
  - Second term in momentum equation is negligibly small
- Turbulence intensities of the order of the velocity defect $\rightarrow u$ and $\Delta U_s$ same order
- Both the turbulence intensities and the velocity defect are small relative to the mean velocity $\rightarrow$ second term on RHS small
- Momentum equation for turbulent wakes far from an obstacle reduces to
Other free-shear flows

- A number of free shear flows can be analyzed in the same way as described for the plane jet and the plane wake:
  - Axisymmetric jets
  - Axisymmetric wakes
  - Free shear layers
  - Thermal plumes
  - Self-propelled wakes

- These types of flow will just be quickly defined here, while further analysis can be found in the literature:
  - Tennekes and Lumley: *A first course in turbulence*, MIT Press, 1972
The plane thermal plume / buoyant plumes

• Upward jet of heated fluid which is driven by the density difference
  – Produced by a body that is hotter than the surrounding fluid
  – In a medium that expands on heating

• Can be analyzed the same way as wakes and jets

• Similar case:
  Liquid of a certain density is poured into a liquid of lower density,

→ upside-down density-driven plume forms

• Influenced by gravity

• Reynolds number increases with streamwise distance

→ Flow will eventually reach a state of full similarity, where all of the mean and turbulent quantities collapse.
Axisymmetric wakes

• Develops behind axisymmetric obstacle

• Structure is not very different from that of the plane wake

• Exception: Reynolds number steadily decreases as the flow develops downstream

• Mean velocity profiles self-similar (if the Reynolds number is large enough to neglect viscous terms in the mean momentum equation)

• Strong three-dimensional structures appear!
Wall-bound turbulent flows

- Turbulence can develop without density variations
- Vorticity generated at the leading edge of a body
  - Will then be diffused, transported and amplified
- Incoming flow has to come to rest at the surface of a body (No-slip-condition)
  - Velocity component tangential to the surface has to be equal to the tangential velocity of the wall
  - i.e. if the body is at rest, the tangential velocity component is zero at the wall
- Kinematic boundary condition
  - Wall-normal velocity component of the fluid has to equal the normal velocity of the surface
    - Normal velocity components in wall-bound flows much smaller than in free shear flows
      - Lower entrainment rate than in free shear flows
Wall-bound turbulent flows

• The idea of the boundary layer saved the discipline of fluid dynamics from
  – d'Alembert's paradox: there seems to be no drag when considering inviscid solutions
  – To fulfill the viscous no-slip condition, at least one viscous stress term has to remain in the governing equations
    – As a consequence, there must be at least two length scales in the flow (unlike for free shear flows where this is not always the case):
      • One describing changes in the mean-flow direction parallel to the surface
      • One characterizing changes in the wall-normal direction

→ Influences estimations for relative importance of terms in governing equations

• There will be different sets of equations for the inner and outer regions of a boundary layer
Wall-bound turbulent flows

- Presence of solid wall imposes constraints that are absent in free shear flows

- Viscosity of the fluid enforces the no-slip condition:
  Velocity of the fluid at a solid surface must be equal to the velocity of the surface

  → Viscosity-dominated characteristic length scale becomes relevant:
    Order of magnitude: $O(v/u)$ ... Viscosity / Scale for turbulent velocity fluctuations

- At high Reynolds numbers, the thickness $\delta$ of the wall-bound flow is much larger than $v/u$

  → Viscosity length scale not relevant for dynamics of entire flow, but only the region in
    immediate vicinity of the surface
    → Two different length scales are relevant

- Further assumptions:
  - Slow downstream evolution of the flow (as in free shear flows)
  - “Thin” shear layer: $\delta/L \ll 1$, i.e. width of layer much smaller than length
Turbulent channel flow

- **Channel flow**: Turbulent flow of incompressible fluid between two parallel plates
  - Distance between the plates: 2h
  - Assumption: plates are infinitely long and wide

- Parallel walls $\rightarrow$ Geometry inhibits the continuing growth of thickness of the flow
  - Fully developed flow (i.e. for sufficiently long pipes or channels):
    - Velocity profile independent of downstream distance $x$
      $\rightarrow$ Nonlinear inertia terms in governing equations suppressed

- The pressure gradient drives the flow against the shear stresses at the walls
Turbulent channel flow

- Kinematic boundary condition: y-component $V$ of the mean velocity is zero at both walls → Continuity equation requires the y-component $V$ to be zero everywhere

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0$$

- Mean momentum equations for fully developed turbulent channel flow:

$$\begin{align*}
U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} &= -\frac{1}{\rho} \frac{\partial P}{\partial x} - \frac{\partial \langle u^2 \rangle}{\partial x} - \frac{\partial \langle uv \rangle}{\partial y} + \nu \frac{\partial^2 U}{\partial x^2} + \nu \frac{\partial^2 U}{\partial y^2} \\
U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} &= -\frac{1}{\rho} \frac{\partial P}{\partial x} - \frac{\partial \langle wu \rangle}{\partial x} - \frac{\partial \langle v^2 \rangle}{\partial y} + \nu \frac{\partial^2 V}{\partial x^2} + \nu \frac{\partial^2 V}{\partial y^2}
\end{align*}$$
Turbulent channel flow

- Mean momentum equations for fully developed turbulent channel flow:

\[
0 = -\frac{1}{\rho} \frac{dP}{dx} + \frac{d}{dy} \left[ -\langle uv \rangle + \nu \frac{dU}{dy} \right]
\]

- Steady flow was assumed \(\Rightarrow\) \(<v^2>\) is independent of \(x\)
  \(\rightarrow\) \(dP/dy = dP_0/dx\)

- Integration of the streamwise component of the momentum equation \(\int_{y=0} dy\) :

\[
\frac{u_*^2}{\rho u_*} = \left[ -\langle uv \rangle + \nu \frac{\partial U}{\partial y} \right] - \frac{y dP_\infty}{\rho d x}
\]

\(\rho u_*\) ... Stress at the surface
\(u_*\) ... Friction velocity
Turbulent channel flow

$u_*^2 = \left[ -\langle uv \rangle + \nu \frac{\partial U}{\partial y} \right] - \frac{y}{\rho} \frac{dP_\infty}{dx}$

- Stress at the surface
- Friction velocity

\( \rho \cdot u_* \) ... Stress at the surface

\( u_* \) ... Friction velocity

- At the surface (\( y=0, y=2h \)):
  - The turbulent velocity fluctuations have to satisfy the no-slip condition
  - Reynolds stress is zero
  - Surface stress is purely viscous stress

- At the center of the channel (\( y=h \)):
  - Shear stress (\( -\rho \langle uv \rangle + \mu \cdot dU/dy \)) must be zero for reasons of symmetry

\[ u_*^2 = -\frac{h}{\rho} \frac{dP_\infty}{dx} \]

Force exerted on the overall flow due to the streamwise pressure gradient is exactly balanced by the wall shear stress.

i.e. shear stress at the wall is determined by the channel width and the pressure gradient only.

Experimental analysis of turbulent flows
Turbulent channel flow

• Rearrangement of the equations:

\[
\frac{u^2}{*} = \left[ -\langle uv \rangle + \nu \frac{\partial U}{\partial y} \right] - \frac{y dP_\infty}{\rho dx}
\]

\[
\frac{u^2}{*} = -\frac{h dP_\infty}{\rho dx}
\]

\[
\rightarrow u^2_*(1 - \frac{y}{h}) = -\langle uv \rangle + \nu \frac{dU}{dy}
\]

• The relative importance of the terms in the mean governing equations depend strongly on the distance from the channel walls:
  – Close to the walls: stress is purely viscous
  – In the core flow: viscous terms negligible

• Channel flow is split up in two regions with different governing equations:
  – Surface layer
  – Core region

• Definition of “inner variables” based on friction velocity:

\[
y^+ = y u_*/\nu, \quad U^+ = U/u_*
\]
Turbulent channel flow

- Applied scaling factors:
  - Friction velocity $u_*$ for the turbulent velocity
    $\rightarrow u_*^2$ for Reynolds shear stresses
  - Half channel height $h$ for changes in the wall-normal direction ($\sim$ thickness of flow)
    $\rightarrow dU/dy$ is scaled with $u_*/h$

- Core region – appropriate non-dimensionalization of governing equations:
  $$\frac{-\langle uv \rangle}{u_*^2} + \frac{\nu}{h u_*} \frac{d(U/u_*)}{d(y/h)} = 1 - \left(\frac{y}{h}\right)$$  
  $y/h = \eta$

  $\rightarrow$ For high Reynolds numbers, the viscous term vanishes.

- Surface layer – non-dimensionalized so that viscous term does not vanish for high Re:
  $$\frac{-\langle uv \rangle}{u_*^2} + \frac{d(U/u_*)}{d(yu_*/\nu)} = 1 - \frac{\nu}{h u_*} \frac{yu_*}{\nu}$$

  - $Re_* = yu_*/\nu$ introduced in scale for $y$
Turbulent channel flow

- Core region – in inner variables:
  \[
  - \frac{<uv>}{u_*^2} + \frac{\nu}{u_* h} \frac{d}{d\eta} \left( \frac{U}{u_*} \right) = 1 - \eta
  \]

- Surface layer – in inner variables:
  \[
  - \frac{<uv>}{u_*^2} + \frac{1}{d} \left( \frac{U}{u_*} \right) = 1 - \frac{\nu}{h u_*} y^+
  \]

  - For smooth surfaces:
    - No additional parameters (i.e. roughness height)
      \[ \rightarrow \text{Solutions for governing equations only a function of distance from the wall} \]
    \[
    \frac{U}{u_*} = f(y^+)
    \]
    and
    \[
    - \frac{<uv>}{u_*^2} = f(y^+)
    \]

    - Boundary conditions: Velocities and fluctuations have to be zero at the wall:
      \[ f(0) = 0, g(0) = 0 \]
Turbulent channel flow – core region

• What do we know about a solution valid in the core region?
  – Mean momentum equation:
    \[
    - \frac{\langle uv \rangle}{u_\|^2} = 1 - \eta
    \]
    \[\rightarrow\text{No information on mean velocity } U\]

  – Turbulent energy budget (see Hinze or Tennekes & Lumley for derivation):
    \[
    - \langle uv \rangle \frac{dU}{dy} = \varepsilon + \frac{d}{dy} \left( \frac{1}{\rho} \langle v \rangle p + \frac{1}{2} q^2 \langle v \rangle \right)
    \]
    Production \hspace{1cm} Viscous dissipation \hspace{1cm} Transport of kinetic energy by turb. motion

  \[\rightarrow\text{Contains } U \text{ explicitly}\]

• Assessment of orders of magnitude for the core region:
  • Reynolds stress \(\langle uv \rangle\) is of order of \(u_\|^2\) (from mean momentum equation)
  • Reynolds stress generates the turbulent energy \(\rightarrow p/\rho\) and \(q^2\) are of same order
  • Large turbulent eddies scale with width of the flow (i.e. channel height)
    \[\rightarrow dy = O(h)\]
    \[\rightarrow dU/dy = O(u_\|/h)\]
Turbulent channel flow – core region

• If no other length scales are of importance (i.e. location sufficiently far away from wall), the velocity gradient can be expressed in form of a (unknown) function $F$:

\[
\frac{dU}{dy} = \frac{u_\ast}{h} \frac{dF}{d\eta}
\]

• Integration of this expression from the center of the channel to the wall:

\[
\frac{U - U_0}{u_\ast} = F(\eta)
\]

$U_0$ ... mean velocity at channel centerline

→ Similarity law for the channel core region has the form of a velocity-defect law

• Flow in the surface layer and core region follows two different sets of equations. → What happens at the interface of these flow regions?
Turbulent channel flow – overlap region

- In the region of overlap between the surface layer and the overlap region, there is of course a steady transition between the layers

→ Velocity gradients have to match:

**Core region:**
\[
\frac{dU}{dy} = \frac{u_*}{h} \frac{dF}{d\eta}
\]

**Surface layer:**
\[
\frac{dU}{dy} = \frac{u_*^2}{\nu} \frac{df}{dy^+}
\]

\[
\frac{u_* \cdot y}{\nu} \cdot \frac{df}{dy^+} = \left(\frac{y}{h}\right) \frac{dF}{d\eta}
\]

\[
= y^+ \quad \text{only a function of } y^+
\]

\[
= \eta \quad \text{only a function of } \eta
\]

→ Both sides of the equation must be equal to the same constant:

\[
1/\kappa \quad \text{with von Kármán constant } \kappa
\]
Turbulent channel flow – overlap region

• Integration of both sides of the equation yields:

\[ F(\eta) = \frac{1}{\kappa} \cdot \ln(\eta) + \text{const.} \]

\[ f(y_+) = \frac{1}{\kappa} \cdot \ln(y_+) + \text{const.} \]

\begin{itemize}
  \item Logarithmic velocity profile
\end{itemize}

• Also the Reynolds-stress distribution has to be steady:

  \begin{itemize}
    \item Core region:
      \[ -\frac{\langle uv \rangle}{u_*^2} = 1 - \eta \]
      \text{for } \eta \to 0 : \quad \left[ -\frac{\langle uv \rangle}{u_*^2} \to 1 \right]
      \text{(Mean momentum equation)}

    \item Surface layer:
      \[ -\frac{\langle uv \rangle}{u_*^2} + \frac{d}{dy^+} \left( \frac{U}{u_*} \right) = 1 \]
      \text{and} \quad \left[ y_+ \cdot \frac{df}{dy^+} = \frac{1}{\kappa} \right] \quad (**)
\end{itemize}

Both equations have to be satisfied.
Turbulent channel flow – overlap region

- Surface layer (continued):

\[
- \frac{\langle uv \rangle}{u_*^2} + \frac{d}{dy^+} \left( \frac{U}{u_*} \right) = 1
\]

\[
y^+ \cdot \frac{df}{dy^+} = \frac{1}{\kappa} \tag{**}
\]

introduce

\[
- \frac{\langle uv \rangle}{u_*^2} = 1 - \frac{1}{\kappa \cdot y^+}
\]

for \( y^+ \to \infty : \)

\[
- \frac{\langle uv \rangle}{u_*^2} \to 1
\]

- The overlap region is thus a region of approximately constant Reynolds shear stress

- The viscous stress is proportional to the second term
  \( \to \) very small compared to the Reynolds stress for \( y^+ >> 1 \)

- Because of this local absence of viscous effects, the overlap region is called **inertial sublayer**.
Turbulent channel flow – overlap region

- The velocity $U_0$ on the channel center line can be determined by combining the discussed equations:

  **Law of the wall**
  \[
  \frac{U}{u_*} = f(y^+) = \frac{1}{\kappa} \cdot \ln (y^+) + \text{const.}
  \]

  **Velocity defect law**
  \[
  \frac{U - U_0}{u_*} = F(\eta) = \frac{1}{\kappa} \cdot \ln (\eta) + \text{const.}
  \]

  **Velocity profile in overlap region**
  \[
  \frac{U}{u_*} = \frac{1}{\kappa} \cdot \ln y^+ + C_1
  \]
  \[
  \frac{U - U_0}{u_*} = \frac{1}{\kappa} \cdot \ln \eta + C_2
  \]

- Both equations valid simultaneously in the inertial sublayer (and are only valid there)

  \[
  \frac{U_0}{u_*} = \frac{1}{\kappa} \cdot \ln \left( \frac{u_* \cdot h}{\nu} \right) + C_1 - C_2
  \]

*Logarithmic friction law*

*Reynolds number based on channel half-height*
Turbulent channel flow – surface layer

- What more do we know about the flow very close to the surface?
  
  - Law of the wall:
    \[
    \frac{U}{u_*} = f(y^+) \quad \text{and} \quad -\frac{<uv>}{u_*^2} = f(y^+)
    \]

- At the surface itself, the total stress is entirely composed of stress is viscous stress.
  
  - If the surface is smooth, this is also true for a thin layer close to the surface.

- Reynolds stress in the inertial sublayer: \( u_*^2 \)

- Mean-velocity gradient in the inertial sublayer: \( dU/dy = u_*/(\kappa \cdot y) \)

  \[ \rightarrow \text{Turbulence production rate} = <uv> \cdot dU/dy \text{ is given by: } u_*^3/(\kappa \cdot y) \]

- In the inertial sublayer, turbulence production is mainly balanced by viscous dissipation \( \varepsilon \)

  \[ \rightarrow \varepsilon = u_*^3/(\kappa \cdot y) \]

- The Kolmogorov microscale is thus:
  \[ \eta = \left( \frac{\nu^3}{\varepsilon} \right)^{\frac{1}{4}} = \left( \frac{\kappa \cdot y \cdot \nu^3}{u_*^3} \right)^{\frac{1}{4}} = \kappa^{\frac{1}{4}} \cdot y \cdot (y^+)^{-\frac{3}{4}} \]

(length scale for small turbulent scales)
Experimental analysis of turbulent flows

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Turbulent channel flow – surface layer

- The integral scale $l$ of the turbulence is of order $y$:
  The largest eddies scale with the thickness of the flow i.e. here the distance from the wall

- The velocity gradient in the inertial sublayer is:
  \[ \frac{dU}{dy} = \frac{U_*}{\kappa \cdot y} \quad \rightarrow \quad l \approx \kappa \cdot y \]

- These two length scales (for the largest and smallest turbulent eddies) can be made dimensionless:
  \[ \eta_+ = \eta \cdot \frac{U_*}{\nu} = \left( \frac{\kappa}{y_+} \right)^{1/4} \quad \text{and} \quad l_+ = \frac{l \cdot U_*}{\nu} = \kappa \cdot y_+ \]

- The thin layer close to the surface where $\eta_+$ is the characteristic length scale is called the **viscous sublayer**.
Turbulent channel flow – surface layer

- Plotting the discussed velocity relations yields a typical distribution for the wall-near zone:
  - Viscous sublayer: $y_+ \leq 5$
    - i.e. Reynolds stress / velocity fluctuations only small contribution to total stress; viscosity dominates
  - Values / constants empirically determined (experimental studies)
  - Velocity fluctuations are still present

- Velocity distribution: linear in viscous sublayer, logarithmic in inertial sublayer
Turbulent channel flow – surface layer

- Composition of the total stress across the wall-near layers:
  - Viscous sublayer: stress is viscosity-dominated
  - Inertial sublayer: constant Reynolds shear-stress, viscous terms negligible
  - Buffer layer: region where neither one of the stresses can be neglected
Turbulent channel flow – surface layer

• The flow of energy in(to) the surface layer:
  – From the point of view of the mean flow, the surface layer is a "sink" for momentum and kinetic energy:
  – Reynolds stresses transfer kinetic energy from the mean-flow into the surface layer
  – In the surface layer, this energy is converted into
    – Turbulent kinetic energy (turbulence production) and
    – Heat (viscous dissipation)

→ Direct loss to viscous dissipation occurs primarily in the viscous sublayer.

• The larger part of the mean-flow kinetic energy transported into the surface layer is used for the maintenance of turbulent kinetic energy.

• Most of the turbulence production occurs in the surface layer

• Core region: not much turbulence production, but transport of kinetic energy into the surface layer
Typical geometries for turbulent flows in engineering

• Free shear flows:
  – Jets
    – Plane jets
    – Round jet (quiet or moving surroundings)
    – “Half jet” – boundary between two parallel flows with different mean velocities
    – Axial / coaxial jets (two-dimensional, rotationally symmetric)
  – Wake flows
    – Plane wake behind cylindrical rod
    – Round wake behind axisymmetric body

• Wall-bound flows:
  – Channel flow
  – Pipe flow
  – Boundary layers
  – Pressure gradient boundary layers

• Complex flow fields:
  – Shock wave / turbulent boundary layer interactions
Turbulent pipe flow

- Axisymmetric parallel flow in a circular pipe of constant diameter

- Assumption: fully developed flow → independent of streamwise direction x

- Greater practical importance than plane channel flow

- Mean momentum equations for fully developed turbulent pipe flow:

\[
- \langle uv \rangle + \nu \frac{dU}{dy} = u_*^2 \left( 1 - \frac{y}{r} \right)
\]

- Conclusions obtained for channel flow apply equally to pipe flow

For details see:
Tennekes and Lumley, A first course in turbulence, MIT Press, 1972
Turbulent pipe flow

- In real pipe flows, the logarithmic velocity profile usually represents the actual velocity profile quite well all through the pipe.

- The width of the inertial sublayer increases with $R^*$.

- Difference between actual velocity profile in the core region and the logarithmic law:
  - Wake function $W(\eta)$
Turbulent pipe flow

• Typical orders of magnitude for the turbulence intensities in pipe flows:
  
  • Turbulence intensity drops slowly when moving from the surface-near region towards the center.
  
  • Approximate values from different experimental data:
    
    • Inertial sublayer:
      – \( u' \approx 2 \cdot u_* \), \( v' \approx 0.8 \cdot u_* \), \( w' \approx 1.4 \cdot u_* \),
      – \( \langle uv \rangle = u_*^2 \approx 0.4 \cdot u'v' \),
      – \( u/U = O(0.01) \)
    
    • Core region:
      \( u' = v' = w' \approx 0.8 \cdot u_* \)

  → The amplitude of the fluctuating velocity component \( v' \) is nearly constant across the pipe.

[Tennekes and Lumley, A first course in turbulence, MIT Press, 1972]
Turbulent pipe flow – rough surface

- For small roughness heights $k$, the roughness does not affect the velocity-defect law, otherwise modifications are required.

- Two characteristic lengths are relevant in the surface layer over a rough wall:
  - Roughness height $k$
  - $\nu/u_*$

- The ratio of those characteristic lengths is the roughness Reynolds number $R_k$:
  \[
  R_k = \frac{k \cdot u_*}{\nu}
  \]

- The law of the wall is therefore now not only a function of $y_+$, but also of $R_k$:
  \[
  \frac{U}{u_*} = f_1(y_+, R_k)
  \quad \text{or} \quad
  \frac{U}{u_*} = f_2\left(\frac{y}{k}, R_k\right)
  \]

- This expression must be matched with the velocity-defect law:
  - Effect of roughness on the logarithmic velocity profile in the inertial sublayer appears as an additive function.

Independent of roughness as long as $k/h \ll 1$
Turbulent pipe flow – rough surface

- Logarithmic velocity profile in the inertial sublayer for rough surfaces:

$$\frac{U}{u_*} = \frac{1}{\kappa} \cdot \ln y_+ + f_3(R_k)$$  
or  
$$\frac{U}{u_*} = \frac{1}{\kappa} \cdot \ln \frac{y}{\kappa} + f_4(R_k)$$

- For small roughness heights \( k, R_k \to 0 \)
  - \( f_3 \) has to become equal to 5 (empirically determined value)
  - \( R_k < 5 : \) no roughness effect on the mean velocity profile in the inertial sublayer
    - Roughness elements stay within the viscous sublayer, where no Reynolds stresses can be generated

- For large \( R_k \), the mean equation of motion is [see Hinze or Tennekes & Lumley for derivation]:

$$-\frac{\langle uv \rangle}{u_*^2} + \frac{1}{R_k} \cdot \frac{d(U/u_*)}{d(y/k)} = 1 - \frac{y}{k} \cdot \frac{k}{r}$$

- A distinct surface layer only exists for small \( k/r \)
- \( R_k > 30 : \) roughness elements generate turbulent wake \( \to f_4(R_k) \) becomes constant
- \( 5 < R_k < 30 : \) constant in the logarithmic part of the velocity profile depends on \( R_k \)

Often, the position \( y = 0 \) is not known very accurately; the additive constant is then absorbed in the definition of \( k \)
Turbulent boundary layers

• Boundary-layer thickness increases in the downstream direction:
  - Reason: loss of momentum at the wall is diffused
    → either by viscosity (molecular mixing) or by turbulent mixing
  - Turbulent boundary layers grow more rapidly than laminar boundary layers

• No general solution for equations of motion
  → Often semi-empirical methods are used to predict the development of a turbulent boundary layer on a specific body.

• Steady plane flow (over smooth surface):
  - Velocity profiles made dimensionless with an appropriate velocity-defect law
    → independent of the Reynolds number and the downstream distance x
  - Equilibrium boundary layers (equivalent to laminar Falkner-Skan boundary layers)

- **Blasius boundary layer** – steady two-dimensional laminar boundary layer forming on a semi-infinite plate parallel to a constant unidirectional flow (self-similar in the x-direction)
- **Falkner–Skan boundary layer** – boundary layer forming on wedge at an angle of attack in a uniform flow field, i.e. boundary layer under pressure gradient (generalization of the Blasius boundary layer)
Turbulent boundary layers

- Boundary layer in steady, incompressible, plane flow over a smooth surface without heat or mass transfer.

![Diagram of boundary layer with various equations and variables.]

Underlying governing equations:

\[
\begin{align*}
U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} & = -\frac{1}{\rho} \frac{\partial P}{\partial x} - \frac{\partial \langle u^2 \rangle}{\partial x} - \frac{\partial \langle uv \rangle}{\partial y} + \nu \frac{\partial^2 U}{\partial x^2} + \nu \frac{\partial^2 U}{\partial y^2} \\
U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} & = -\frac{1}{\rho} \frac{\partial P}{\partial y} - \frac{\partial \langle uv \rangle}{\partial x} - \frac{\partial \langle v^2 \rangle}{\partial y} + \nu \frac{\partial^2 V}{\partial x^2} + \nu \frac{\partial^2 V}{\partial y^2} \\
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} & = 0
\end{align*}
\]
Turbulent boundary layers

- Length scale $L$ – associated with rate of change of $U_0$ downstream:

$$\frac{1}{L} = \left| \frac{1}{U_0} \frac{\partial U_0}{\partial x} \right| = 0$$

- For uniform outer flow: $L \to \infty$

→ Appropriate length scale: distance $x$ from a suitably defined origin

- Solutions to the equations of motion that satisfy a velocity-defect law are searched:

$$\frac{U - U_0}{u_*} = F \left( \frac{y}{\delta} \right)$$

→ Self-preserving flow

- Idea: Find a self-preserving solution that is independent of Reynolds number
  - Describes entire family of flows
  - Non-dimensionalized pressure gradient is the only parameter

Not valid in the surface layer. Has to be treated separately.
Turbulent boundary layers

- Procedure to determine approximations for governing equations same as for wake flow

- Typical length scales:
  - Streamwise distance $L$ from origin for changes in the main $x$-direction
  - Thickness $\delta$ of the boundary layer for changes in the wall-normal direction

- The necessity to keep at least one turbulence term in the equations gives some information about the relative order of magnitude of different scales:

$$\frac{\delta}{L} \sim \frac{u^2}{U_s \Delta U_s}$$

- Unfortunately, this assumption leads to the conclusion that the viscous terms are negligible (for high Reynolds numbers)
Turbulent boundary layers

• Resulting components of the mean momentum equation:

\[ U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = -\frac{dP_\infty}{dx} - \frac{\partial}{\partial y} \langle uv \rangle - \frac{\partial}{\partial x} \left[ \langle u^2 \rangle - \langle v^2 \rangle \right] \]

\[ 0 \approx -\frac{1}{\rho} \frac{\partial P}{\partial y} - \frac{\partial \langle v^2 \rangle}{\partial y} \]

• Substitution and integration from wall to location x:

\[ U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = -\frac{1}{\rho} \frac{\partial P}{\partial x} - \frac{\partial \langle uv \rangle}{\partial y} - \frac{\partial \langle u^2 \rangle}{\partial x} \]

→ Order of magnitude analysis for a boundary layer yields same equations as for a free shear layer.

→ To satisfy the no-slip condition, a sublayer within the boundary layer must exist, where a viscous term is relevant.

• The relations discussed so far are only valid in the outer layer of the boundary layer.
Turbulent boundary layers – the inner layer

- No-slip condition has to be satisfied → viscous term remains in the governing equations → Mean velocity near the wall has to change rapidly enough

→ New length scale $\eta$ for changes in the $y$-direction very near the wall: $\eta \ll \delta$

- Scaling of mean and turbulence velocities near the wall
  - Very close to the wall → velocity has dropped considerably (no-slip condition) → scaling with outer-flow velocity $U_0$ not sensible
  - New “wall-scaling” $u_w$ with $u_w \ll U_0$
    - Turbulence intensity near the wall is relatively high
      → Same scaling can be used for mean and turbulent velocities

- Length scale $L$ to characterize changes in the $x$-direction
  - Variation even less rapid than in the outer boundary layer (due to proximity to wall)
Turbulent boundary layers – the inner layer

- Estimates for x-momentum equation

\[
\frac{U}{\partial x} + \frac{V}{\partial y} = \frac{1}{\rho} \frac{\partial P}{\partial x} - \frac{\partial \langle u^2 \rangle}{\partial x} - \frac{\partial \langle uv \rangle}{\partial y} + \nu \frac{\partial^2 U}{\partial x^2} + \nu \frac{\partial^2 U}{\partial y^2}
\]

From continuity equation

- Which terms will remain?

- No-slip condition has to be satisfied \(\rightarrow\) at least one viscous term has to remain
  - Scale of largest viscous term used to normalize other terms

Experimental analysis of turbulent flows

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Turbulent boundary layers – the inner layer

- Estimates for $x$-momentum equation

$$
\frac{U}{\nu} \frac{\partial U}{\partial x} + \frac{V}{\nu} \frac{\partial U}{\partial y} \left( \frac{u_w \eta}{\nu} \right) \frac{\eta}{L} + \left( \frac{u_w \eta}{\nu} \right) \frac{\eta}{L} = -\frac{1}{\rho} \frac{\partial P}{\partial x} - \frac{\partial \langle u^2 \rangle}{\partial x} - \frac{\partial \langle u w \rangle}{\partial y} + \nu \frac{\partial^2 U}{\partial x^2} + \nu \frac{\partial^2 U}{\partial y^2} + \frac{u_w \eta}{\nu} \frac{\eta^2}{L^2} + \frac{u_w \eta}{\nu} O(1)
$$

- Whether or not the Reynolds stress term remains depends on the choice of $\eta$:

$$\frac{u_w \cdot \eta}{\nu} = O(1) \quad \text{or} \quad \frac{u_w \cdot \eta}{\nu} \rightarrow 0$$

- Typically, $\eta$ is chosen in a way that the Reynolds stress term remains
Turbulent boundary layers – the inner layer

- Estimates for x-momentum equation

\[
\frac{U}{\partial x} + \frac{V}{\partial y} = \frac{u_w \cdot \eta}{\nu} = O(1)
\]
\[
\eta \ll L \ldots \text{inner layer}
\]

\[
\begin{align*}
\frac{1}{\rho} \frac{\partial P}{\partial x} - \frac{\partial \langle u^2 \rangle}{\partial x} - \frac{\partial \langle uv \rangle}{\partial y} + \nu \frac{\partial^2 U}{\partial x^2} + \nu \frac{\partial^2 U}{\partial y^2} \\
\end{align*}
\]

- Same procedure applied to the y-momentum equation
  → pressure is imposed from outer flow

- In first order, the mean momentum equation for the near wall region reduces to:

\[
0 \approx \frac{\partial}{\partial y} \left[ -\langle uv \rangle + \nu \frac{\partial U}{\partial y} \right]
\]
Turbulent boundary layers – the inner layer

• Integration of the mean momentum equation from the wall to location $y$:

$$0 \approx \frac{\partial}{\partial y} \left[ -\langle uv \rangle + \nu \frac{\partial U}{\partial y} \right]$$

$$0 = -\langle uv \rangle - \langle uv \rangle |_{y=0} + \nu \frac{\partial U}{\partial y} - \nu \frac{\partial U}{\partial y} |_{y=0}$$

no-slip boundary condition at the wall

• Wall shear stress:

$$\tau_w \equiv \mu \frac{\partial U}{\partial y} |_{y=0}$$

$$\frac{\tau_w}{\rho} = -\langle uv \rangle + \nu \frac{\partial U}{\partial y}$$

• This means that the total stress in the wall layer is constant (for $Re \to \infty$)
  $\rightarrow$ Wall layer is also called Constant Stress Layer

• In reality, for large (but finite) Reynolds numbers the total stress is only almost constant.
Turbulent boundary layers – the inner layer

• Appropriate scales for the wall region of the boundary layer:
  – Inner velocity scale: Friction velocity \( u_\ast \)
    \[
    u_\ast^2 \equiv \frac{\tau_w}{\rho}
    \]
  – Inner length scale: \( \eta = \nu/u_\ast \)

• The inner length scale is used to define dimensionless inner variables:
    \[
    y^+ \equiv \frac{y}{\eta} = \frac{yu_\ast}{\nu}
    \]

• Typically, turbulent moments are plotted in inner variables in the wall region
  = made dimensionless with \( u_\ast \) and plotted versus \( y^+ \)

• In analogy, non-dimensionalized variables in the outer layer are called outer variables:
    \[
    \bar{y} = \frac{y}{\delta}
    \]

• Inner variables are usually necessary for \( \bar{y} < 0.1 \), outer variables for \( y^+ > 30 \).
Turbulent boundary layers – the inner layer

• The inner Reynolds number is approximately unity:

\[ \frac{u^* \cdot \eta}{\nu} \approx 1 \]

→ Viscous and inertial terms are of about the same magnitude.

• The inner layer is also called **viscous sublayer**
  – in contrast to the outer layer where the mean flow is dominated by the Reynolds stress alone.

• In the immediate vicinity of the wall, the Reynolds stress becomes less important than the viscous stress.
  – For \( y^* < 3 \) the Reynolds stress term is negligible.
  – In this area, the governing equation reduces to:

\[ u^2 \approx \nu \frac{\partial U}{\partial y} \]

→

\[ U(x, y) = \frac{u^2 y}{\nu} \]

→ The velocity distribution becomes linear; this region is thus called **linear sublayer**.
Turbulent boundary layers – the inner layer

• The linear sublayer is only a small fraction of the boundary layer and difficult to resolve spatially.

• It has, however, some practical applications in different approaches to find solutions for a flow field:
  • In numerical solutions, it can be used as a boundary condition (if the resolution is good enough to resolve this part of the flow).
  • In experiments, velocity measurements that are resolved beyond $y^+ < 3$ give a good estimate of the wall shear stress.
Turbulent boundary layers with pressure gradient

- Averaged equation of motion for a turbulent boundary layer with pressure gradient:

\[
U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = -\frac{1}{\rho} \frac{dP_\infty}{dx} + \frac{\partial}{\partial y} \left[ -\langle uv \rangle + \nu \frac{\partial U}{\partial y} \right]
\]

- y-momentum equation was integrated to replace the local pressure by pressure \( P_\infty \) imposed from the outer flow

- Pressure gradients – associated with acceleration or deceleration of the mean flow
  - Adverse pressure gradient = static pressure increases in direction of the flow
    → Deceleration of the flow
    → Large enough pressure increase → velocity close to the surface may become zero, or the flow direction can even be reversed
    → Flow separation
    → Pressure distribution along the surface modified
    → Influence on lift and drag

- Favorable pressure gradient = static pressure decreases in the flow direction
  → Acceleration of the flow